Riesz basis of exponentials for convex polytopes with symmetric faces

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based on a joint work with Nir Lev (BIU)

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Introduction

Given a bounded measurable set $\Omega \subset \mathbb{R}^d$ of positive measure, when is it possible to find a countable set of frequencies $\Lambda \subset \mathbb{R}^d$ so that the system

$$E(\Lambda) := \{e_\lambda\}_{\lambda \in \Lambda}, \qquad e_\lambda(x) = e^{2\pi i \langle x, \lambda \rangle}$$

is a basis for $L^2(\Omega)$?

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is a basis for $L^2(\Omega)$?

The answer depends on what we mean by a **basis**!

Orthonormal basis

In a Hilbert space H, orthonormal basis (ONB) are the best type of basis one can expect.

Recall

Great properties: if a system of vectors $\{f_n\}$ is an ONB for H, then for any $f \in H$

$$f = \sum \langle f, f_n \rangle f_n, \qquad \|f\| = \|\langle f, f_n \rangle\|_{\ell^2}.$$

Perfect reconstruction of f via its coefficients $\langle f, f_n \rangle$.

If $E(\Lambda)$ is an orthonormal basis for $L^2(\Omega)$, we say that Λ is a **spectrum** for Ω, and Ω is called a **spectral** set.

Classical example: Fourier basis $E(\mathbb{Z}^d)$ of $L^2([0,1]^d)$.

Orthonormal basis of exponentials

Question

Given an arbitrary set Ω, does there exist an ONB of exponential functions for $L^2(\Omega)$? (Does Ω admit a spectrum Λ?)

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Answer: It depends on the geometry of Ω. More precisely, the answer is intimately related to the concept of **tiling** by translations.

Tiling by translations

Definition

We say that a measurable set $\Omega \subset \mathbb{R}^d$ tiles $\mathbb{R}^d$ by translations if there exists a (discrete) set $T \subset \mathbb{R}^d$ if

$$\bigcup_{t \in T} (\Omega + t) = \mathbb{R}^d,$$

and $|(\Omega + t) \cap (\Omega + t')| = 0$ for every $t, t' \in T$ such that $t \neq t'$.
Equivalently,

$$\sum_{t \in T} \chi_\Omega(x - t) = 1 \quad \text{a.e. } x \in \mathbb{R}^d.$$

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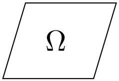
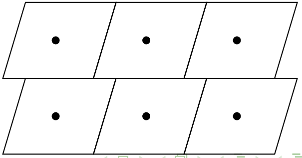
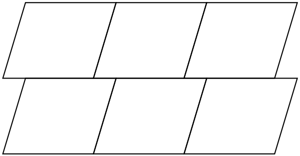




Tiling sets: examples

Sets that **tile** $\mathbb{R}^2$ by translations:

- Parallelogram



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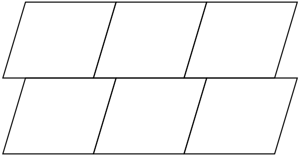

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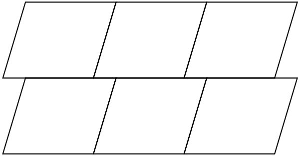

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Sets that **do not** tile $\mathbb{R}^2$ by translations:

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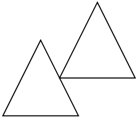

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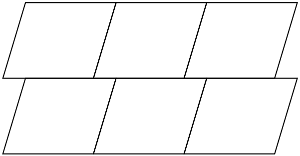

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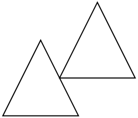
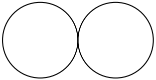


- Circle



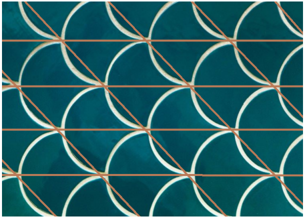
In 1974, B. Fuglede conjectured the following:

Conjecture

$\Omega \subset \mathbb{R}^d$ admits a spectrum if and only if it tiles $\mathbb{R}^d$ by translations.

He obtained partial results towards the conjecture:

- when Λ is a lattice, i.e., $\Lambda = A\mathbb{Z}^d$ for some invertible $d \times d$ matrix A. In this case, $T = (A^T)^{-1}\mathbb{Z}^d$ is the **dual lattice** of Λ, also denoted Λ^*;
- when T is a lattice (and in this case $\Lambda = T^*$ is a spectrum for Ω).



Tiling convex sets

One of the directions of the Fuglede conjecture (tiling $\Rightarrow$ spectral) has been known to be true since long ago for convex sets Ω

Theorem (Venkov, 1954; McMullen, 1980)

If a convex body $\Omega \subset \mathbb{R}^d$ tiles $\mathbb{R}^d$ by translations, then Ω is a centrally symmetric polytope and moreover it tiles $\mathbb{R}^d$ by lattice translations.

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Remark

The statement of Venkov's theorem is much stronger!

The Fuglede conjecture after 2000

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Is Fuglede conjecture true for general sets Ω in $d = 1, 2$?

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Is Fuglede conjecture true for general sets Ω in $d = 1, 2$?

- Many partial results...

Looking for alternatives to exponential ONBs

Theorem (Venkov, 1954; McMullen, 1980)

If a convex body $\Omega \subset \mathbb{R}^d$ tiles $\mathbb{R}^d$ by translations, then Ω is a centrally symmetric polytope and moreover it tiles $\mathbb{R}^d$ by lattice translations.

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Question

Can we find weaker structures than exponential ONBs that will provide a useful decomposition of $L^2(\Omega)$ for a larger class of convex sets Ω?

Riesz bases

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Equivalent definition

A system of vectors $\{f_n\} \subset H$ is a Riesz basis for H if and only if every $f \in H$ admits a representation

$$f = \sum c_n f_n$$

and such that the coefficients $\{c_n\}$ satisfy the relation

$$A\|f\|^2 \leq \sum |c_n|^2 \leq B\|f\|^2,$$

where $0 < A \leq B$ do not depend on f.

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A system of vectors $\{f_n\} \subset H$ is a Riesz basis for H if and only if it satisfies the following three conditions:

1. $\{f_n\}$ is complete in H (i.e., if $\langle f, f_n \rangle = 0$ for all n, then $f \equiv 0$);
2. for every $f \in H$ we have $\sum |\langle f, f_n \rangle|^2 < \infty$;
3. for any sequence $\{c_n\} \in \ell^2$ there exists $f \in H$ such that $\langle f, f_n \rangle = c_n$ for all n.

Riesz bases of exponentials - a simple example

Example (Kadec's 1/4-Theorem)

If $\Lambda := \{\lambda_n\} \subset \mathbb{R}$ is such that

$$|\lambda_n - n| \leq L < \frac{1}{4} \quad \text{for all } n \in \mathbb{Z},$$

then $E(\Lambda) = \{e^{2\pi i \lambda_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$. The constant 1/4 is sharp.

Riesz basis of exponentials: known results

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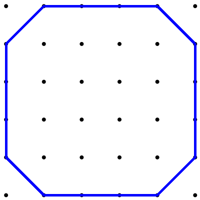

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- 2017, D. Walnut: same conclusion as Y. Lyubarskii and A. Rashkovskii, different approach.

Main result

Theorem (D., Lev)

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Let Ω be a centrally symmetric polytope on $\mathbb{R}^d$, whose faces of all dimensions are centrally symmetric. Then $L^2(\Omega)$ admits a Riesz basis of exponentials $E(\Lambda)$.

The proof is based on a recent approach due to D. Walnut ($d = 2$). We also need the Paley-Wiener stability theorem:

Theorem

Let Ω be a bounded set and let $\Lambda = \{\lambda_n\}$ be a sequence of points such that $E(\Lambda)$ is a Riesz basis for $L^2(\Omega)$. Then there exists a constant $\eta = \eta(\Omega, \Lambda) > 0$ such that if a sequence $\Lambda' = \{\lambda'_n\}$ satisfies

$$|\lambda_n - \lambda'_n| \leq \eta$$

for all n, then $E(\Lambda')$ is also a Riesz basis for $L^2(\Omega)$.

Paley-Wiener spaces of functions

For a bounded measurable set $\Omega \subset \mathbb{R}^d$ of positive measure, the Paley-Wiener space $PW(\Omega)$ is the set of all $F \in L^2(\mathbb{R}^d)$ satisfying

$$F(x) = \int_\Omega f(t) e^{-2\pi i \langle x,t \rangle} \, dt, \quad f \in L^2(\Omega),$$

i.e., the space of Fourier transforms of $f \in L^2(\Omega)$.

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Definition

A set $\Lambda \subset \mathbb{R}^d$ is called a set of **uniqueness** for $PW(\Omega)$ if whenever $F \in PW(\Omega)$ satisfies $F(\lambda) = 0$ for every $\lambda \in \Lambda$, then $F \equiv 0$. In other words, F is uniquely determined by its values in Λ.

Definition

A set $\Lambda \subset \mathbb{R}^d$ is called a set of **interpolation** for $PW(\Omega)$ if for any $\{c_\lambda\} \in \ell^2(\Lambda)$ there exists at least one $F \in PW(\Omega)$ such that $F(\lambda) = c_\lambda$ for all $\lambda \in \Lambda$.

Characterization of Riesz bases of exponentials

The following is well known:

Proposition

$E(\Lambda)$ is a Riesz basis for $L^2(\Omega)$ if and only if Λ is a set of uniqueness and interpolation for $PW(\Omega)$.

The proof of our main result consists in constructing sets Λ of interpolation and uniqueness for $PW(\Omega)$.

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Proposition

If Λ is a set of interpolation for $PW(\Omega)$, then Λ is uniformly discrete.

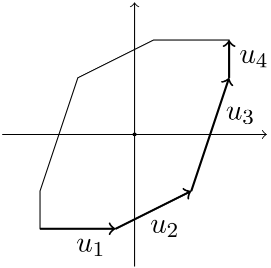
We say that $\Lambda \subset \mathbb{R}^d$ is a **uniformly discrete** set if

$$\inf_{\lambda,\lambda' \in \Lambda} |\lambda - \lambda'| \geq c > 0.$$

Proof of the main result (2 dimensions)

Any centrally symmetric polygon $\Omega_N \subset \mathbb{R}^2$ with $2N$ sides is a Minkowski sum of N vectors $u_1, \ldots, u_N$:

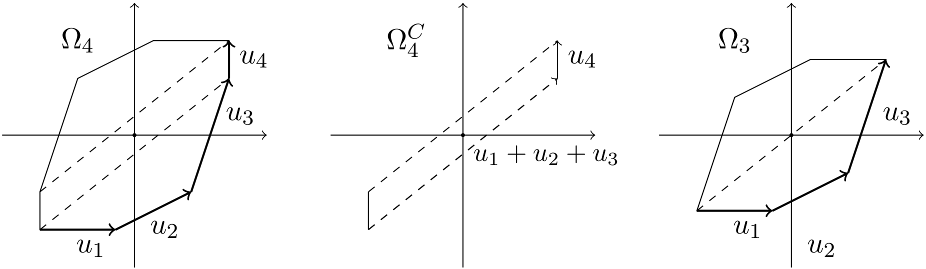
$$\Omega_N = \Omega(u_1, \ldots, u_N) = \left\{ \sum_{k=1}^{N} t_k u_k : -\frac{1}{2} \leq t_k \leq \frac{1}{2},\, k = 1, \ldots, N \right\},$$



Notation

Denote $\Omega_{N-1} = \Omega(u_1, \dots, u_{N-1})$, and Ω_N^C the central parallelogram of Ω_N (given by u_N). Formally,

$$\Omega_N^C = \left\{ t_1 \sum_{k=1}^{N-1} u_k + t_2 u_N : -\frac{1}{2} \le t_1, t_2 \le \frac{1}{2} \right\}.$$

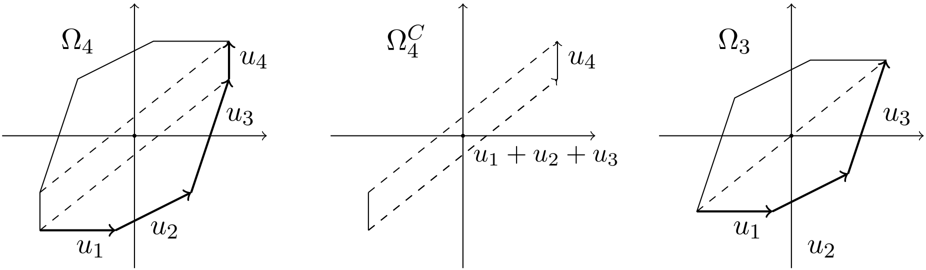


Decomposition lemma

Lemma

For any $F \in PW(\Omega_N)$ there exist functions $G \in PW(\Omega_{N-1})$ and $H \in PW(\Omega_N^C)$ such that

$$F(x) = H(x) + \sin(\pi\langle x, u_N\rangle)G(x)$$



Decomposition lemma - idea of the proof

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For any $F \in PW(\Omega_N)$ *there exist functions* $G \in PW(\Omega_{N-1})$ *and* $H \in PW(\Omega_N^C)$ *such that*

$$F(x) = H(x) + \sin(\pi\langle x, u_N\rangle)G(x)$$

First we write

$$\sin(\pi\langle x, u_N\rangle) = \frac{e^{2\pi i\langle x, \frac{u_N}{2}\rangle} - e^{-2\pi i\langle x, \frac{u_N}{2}\rangle}}{2i}$$

Taking Fourier transforms,

$$\widehat{F}(x) = \widehat{H}(x) + \frac{\widehat{G}(x - \frac{u_N}{2}) - \widehat{G}(x + \frac{u_N}{2})}{2i}$$

Decomposition lemma - idea of the proof II

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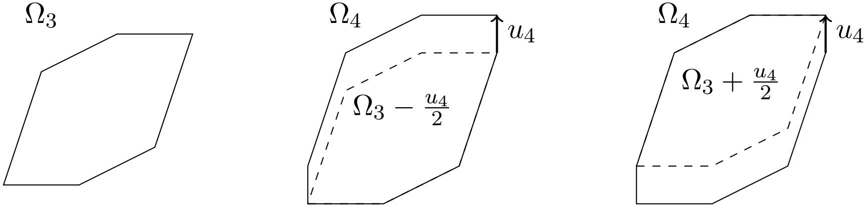

$\widehat{G}$ can be chosen in a way that

$$\widehat{F}(x) = \frac{\widehat{G}(x - \frac{u_N}{2}) - \widehat{G}(x + \frac{u_N}{2})}{2i}, \quad x \in \Omega_N \backslash \Omega_N^C.$$

Construction of the Riesz basis

Recall

Finding a Riesz basis for Ω_N is equivalent to finding a set of uniqueness and interpolation for $PW(\Omega_N)$.

Assume there exists a Riesz basis $E(\Lambda_{N-1})$ for Ω_{N-1}.

$$F(x) = H(x) + \sin(\pi\langle x, u_N\rangle)G(x),$$

$F \in PW(\Omega_N)$, $H \in PW(\Omega_N^C)$, and $G \in PW(\Omega_{N-1})$.

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a set of uniqueness and interpolation for $PW(\Omega_N)$? **Not necessarily!**

We check if $\Lambda_{N-1} \cup \Delta_N$ is a set of uniqueness for $PW(\Omega_N)$. Denote by Z_N the set of zeros of $\sin(\pi\langle x, u_N\rangle)$, and assume $F(\lambda) = 0$ for all $\lambda \in \Lambda_{N-1} \cup \Delta_N$.

Construction of the Riesz basis - set of uniqueness

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- It is easy to check that $\Delta_N \subset Z_N$. If $\delta \in \Delta_N$,

$$F(\delta) = H(\delta) + \sin(\pi\langle \delta, u_N\rangle)G(\delta) = H(\delta) = 0.$$

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Remark

The sets Z_N and Λ_{N-1} must not have common points!

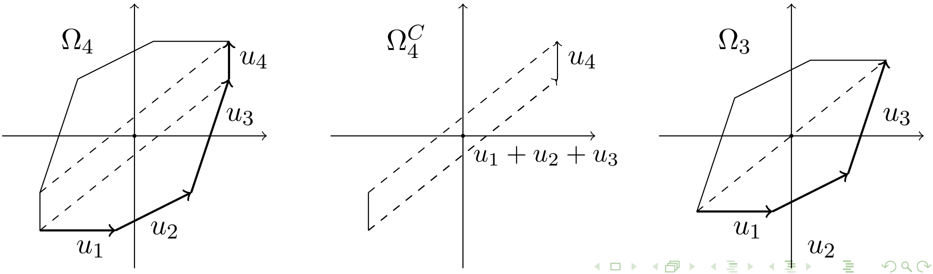
Construction of the Riesz basis - set of uniqueness

Corollary

If Δ_N is a set of uniqueness for $PW(\Omega_N^C)$ and Λ_{N-1} is a set of uniqueness for $PW(\Omega_{N-1})$ such that

$$\{x \in \mathbb{R}^2 : \sin(\pi\langle x, u_N\rangle) = 0\} \cap \Lambda_{N-1} = \emptyset,$$

then $\Delta_N \cup \Lambda_{N-1}$ is a set of uniqueness for $PW(\Omega_N)$.



Remark

In the case of sets of interpolation, the situation is worse. We need the sets Z_N and Λ_{N-1} to be **separated**, i.e.,

$$\inf_{\lambda \in \Lambda_{N-1}} |\sin(\pi\langle\lambda, u_N\rangle)| > 0.$$

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The stability theorem comes into play:

Theorem (Paley-Wiener stability theorem)

Let Ω be a bounded set and let $\Lambda = \{\lambda_n\}$ be a sequence of points such that $E(\Lambda)$ is a Riesz basis for $L^2(\Omega)$. Then there exists a constant $\eta = \eta(\Omega, \Lambda) > 0$ such that if a sequence $\Lambda' = \{\lambda'_n\}$ satisfies

$$|\lambda_n - \lambda'_n| \leq \eta$$

for all n, then $E(\Lambda')$ is also a Riesz basis for $L^2(\Omega)$.

Construction of the Riesz basis - set of interpolation

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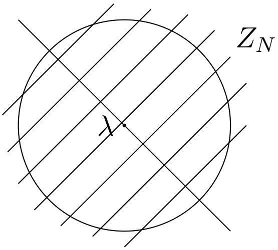
Goal: To slightly perturb the set Λ_{N-1} to obtain a set Λ'_{N-1}, so that

- it is still a set of uniqueness and interpolation for $PW(\Omega_{N-1})$;
- Λ'_{N-1} and Z_N are separated.

Perturbation method

Let $\eta > 0$ be the constant from the Paley-Wiener perturbation theorem. For any $\lambda \in \Lambda_{N-1}$,



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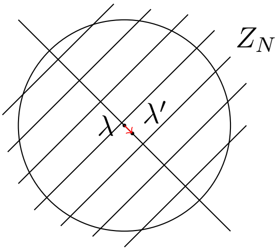

$B(\lambda, \eta)$

Define $\Lambda'_{N-1} = \{\lambda'_n\}$. Then

$$\inf_{\lambda' \in \Lambda'_{N-1}} |\sin(\pi\langle\lambda', u_N\rangle)| \geq c(\eta, \Lambda_{N-1}, u_N) > 0.$$

Construction of the Riesz basis - set of interpolation

Recall

A set $\Lambda \subset \mathbb{R}^d$ is a set of **interpolation** for $PW(\Omega)$ if for any $\{c_\lambda\} \in \ell^2(\Lambda)$ there exists at least one $F \in PW(\Omega)$ such that $F(\lambda) = c_\lambda$ for all $\lambda \in \Lambda$.

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Let Δ_N and Λ_{N-1} be sets of interpolation for $PW(\Omega_N^C)$ and $PW(\Omega_{N-1})$, respectively.

- Perturb Λ_{N-1} to obtain Λ'_{N-1} so that

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Lemma

If $\Omega \subset \mathbb{R}^d$ is bounded and of positive measure, for any ***uniformly discrete*** *set $\Lambda \subset \mathbb{R}^d$ there is a constant $C = C(\Lambda, \Omega)$ such that*

$$\sum_{\lambda\in\Lambda} |H(\lambda)|^2 \leq C\|H\|^2_{L^2(\mathbb{R}^d)}$$

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- We are done!

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- Poor control of "Riesz" constants

$$A\|f\|^2 \leq \sum |c_n|^2 \leq B\|f\|^2.$$

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- Does there exist a (nontrivial) set that does not admit a Riesz basis of exponentials?

Thank you for your attention!