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Twenty-five years of greedy bases

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Abstract

Although the basic idea behind the concept of a greedy basis had been around for some time, the formal development of a theory of greedy bases was initiated in 1999 with the publication of the article [S.V. Konyagin and V.N. Temlyakov, A remark on greedy approximation in Banach spaces, East J. Approx. 5 (3) (1999), 365-379]. The theoretical simplicity of the thresholding greedy algorithm became a model for a procedure widely used in numerical applications and the subject of greedy bases evolved very rapidly from the point of view of approximation theory. The idea of studying greedy bases and related greedy algorithms attracted also the attention of researchers with a classical Banach space theory background. From the more abstract point of functional analysis, the theory of greedy bases and its derivates evolved very fast as many fundamental results were discovered and new ramifications branched out. Hundreds of papers on greedy-like bases and several monographs have been written since the appearance of the aforementioned foundational paper. After twenty-five years, the theory is very much alive and it continues to be a very active research topic both for functional analysts and for researchers interested in the applied nature of nonlinear approximation alike. This is why we believe it is a good moment to gather a selection of 25 open problems (one per year since 1999!) whose solution would contribute to advance the state of art of this beautiful topic.

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1. Introduction

Greedy algorithms provide sparse representations (or approximations) of a given image/signal in terms of a given system of elements of the ambient space. In a mathematical setting, an image or signal is considered to be a function of a Banach space. For instance, a two-dimensional image can be viewed as a function of two variables belonging to the Hilbert space L_2 or, more generally, to the Banach space L_p, $1 \le p \le \infty$. Usually we assume that the system used for representation has some natural properties and we call it a *dictionary*. For an element f from a Banach space $\mathbb{X}$ and a fixed m, we consider approximants which are linear combinations of m terms from a dictionary $\mathcal{D}$. We call such an approximant an *m-term approximant* of f with respect to $\mathcal{D}$.

In sparse approximation, a greedy algorithm is an algorithm that uses a greedy step in searching for a new element to be added to a given m-term approximant. By a *greedy step*, we mean one which maximizes a certain functional determined by information from the previous steps of the algorithm. We obtain different types of greedy algorithms by varying the above-mentioned functional and also by using different ways of constructing (i.e., choosing coefficients of the linear combination) the m-term approximant from m previously selected elements of the dictionary.

A classical problem of mathematical and numerical analysis that goes back to the origins of Taylor's and Fourier's expansions, is to approximately represent a given function from a space. The first step to solve the representation problem is to choose a representation system. Traditionally, a representation system has some natural features such as minimality, or orthogonality, that is, a simple structure which allows nice computational properties. The most typical representation systems are the trigonometric system

$$x \mapsto e^{ikx}, \quad k \in \mathbb{Z},$$

the algebraic system

$$x \mapsto x^k, \quad k \in \mathbb{Z},\ k \ge 0,$$

the spline system, the wavelet system, and their multivariate versions. In general we may speak of a basis $\mathcal{X} = (\boldsymbol{x}_k)_{k=1}^{\infty}$ (in a sense that we will specify below) in a Banach (or quasi-Banach) space $\mathbb{X}$.

The second step to solve the representation problem is to choose the form of the approximant to be built from the chosen representation system $\mathcal{X}$. In a classical way which was used for centuries, given $m \in \mathbb{N}$, an approximant a_m is a polynomial of order m with respect to $\mathcal{X}$,

$$a_m = \sum_{k=1}^{m} c_k\, \boldsymbol{x}_k,$$

for some scalars c_k, $k = 1, \ldots, m$. In numerical analysis and approximation theory it was understood that in many problems from signal/image processing it is more beneficial to use an m-term approximant with respect to $\mathcal{X}$ than a polynomial of order m. This means that for $f \in \mathbb{X}$ we look for an approximant of the form

$$a_m(f) := \sum_{k \in \Lambda(f)} c_k\, \boldsymbol{x}_k,$$

where $\Lambda(f)$ is a set of m indices which is determined by f.

The third step to solve the representation problem is to choose a construction method of the approximant. In linear theory, partial sums of the corresponding expansion of f with respect to the basis $\mathcal{X}$ is a standard method. It turns out that, in nonlinear theory, greedy approximants are natural substitutes for the partial sums.

We emphasize that, although in the beginning this theory developed within the framework of approximation theory, soon after the appearance of [46] the centre of activities in the area moved to functional analysis. Indeed, the introduction of new types of bases and the achievement of their characterizations in terms of classical properties from Banach space theory caught the attention of the specialists, who gave impetus to the theory and set the foundations for a fruitful and novel research topic. As a result we now have different notations for the same objects, which come from approximation theory (see e.g. [58,60]) and functional analysis. In keeping with current usage, in the next section we will present the notation and terminology in the way it is nowadays used in the modern functional analysis approach to the subject, as the reader can find in [19, Chapter 10] or [13]. After the preliminary Section 2, we present a selection of topics that reflect the state of art of the theory and suggest within each section the problems that we believe should be addressed in order to make meaningful advances.

2. Notation and terminology

A *minimal system* in a Banach (or quasi-Banach) space $\mathbb{X}$ over the real or complex field $\mathbb{F}$ will be a sequence $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ in $\mathbb{X}$ for which there is a sequence $\mathcal{X}^* = (\boldsymbol{x}_n^*)_{n=1}^\infty$ in $\mathbb{X}^*$ such that $\boldsymbol{x}_n^*(\boldsymbol{x}_n) = \delta_{n,k}$ for all k and n in $\mathbb{N}$. If $\mathcal{X}$ is *complete*, i.e., its closed linear span $[\mathcal{X}] = [\boldsymbol{x}_n : n \in \mathbb{N}]$ is the entire space $\mathbb{X}$, then $\mathcal{X}^*$ is unique, and we call it the *dual minimal system* of $\mathbb{X}$. In this case, we can associate the biorthogonal system $(\boldsymbol{x}_n, \boldsymbol{x}_n^*)_{n=1}^\infty$ to $\mathcal{X}$. Also, we can define for each finite subset A of $\mathbb{N}$ the *coordinate projection* on A relative to $\mathcal{X}$ as

$$S_A : \mathbb{X} \to \mathbb{X}, \quad f \mapsto \sum_{n\in A} \boldsymbol{x}_n^*(f)\, \boldsymbol{x}_n.$$

A finite subset A of $\mathbb{N}$ is said to be a *greedy set* of $f \in \mathbb{X}$ with respect to the complete minimal system $\mathcal{X}$ if

$$\left|\boldsymbol{x}_n^*(f)\right| \geq \left|\boldsymbol{x}_k^*(f)\right|, \quad n \in A, \quad k \in \mathbb{N} \setminus A,$$

in which case $S_A(f)$ is said to be a *greedy projection*. The map

$$\mathcal{F} : \mathbb{X} \to \mathbb{F}^{\mathbb{N}}, \quad f \mapsto \boldsymbol{x}_n^*(f)$$

will be called the *coefficient transform* with respect to $\mathcal{X}$. The support of a function (or signal) $f \in \mathbb{X}$ with respect to $\mathcal{X}$ is the set

$$\operatorname{supp}(f) = \{n \in \mathbb{N} : \boldsymbol{x}_n^*(f) \neq 0\}.$$

A sequence $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ in $\mathbb{X}$ is said to be a *Schauder basis* if for each $f \in \mathbb{X}$ there is a unique sequence $\alpha(f) = (a_n)_{n=1}^\infty$ in $\mathbb{F}$ such that $f = \sum_{n=1}^\infty a_n\, \boldsymbol{x}_n$. If this series converges unconditionality for all $f \in \mathbb{X}$, $\mathcal{X}$ is said to be an *unconditional basis* of $\mathbb{X}$. It is known that $\mathcal{X}$ is an unconditional basis if and only if it is a complete minimal system for which the coordinate projections are uniformly bounded. If $C \in [1, \infty)$ is such that

$$\|S_A\| \leq C, \quad A \subseteq \mathbb{N}, \ |A| < \infty,$$

we say that $\mathcal{X}$ is C-unconditional. Furthermore, unconditional bases satisfy the estimate

$$\left\| \sum_{n=1}^{\infty} a_n \, \boldsymbol{x}_n \right\| \le C \left\| \sum_{n=1}^{\infty} b_n \, \boldsymbol{x}_n \right\|, \quad |a_n| \le |b_n| . \tag{2.1}$$

If (2.1) holds for a given constant C, we say that the basis is *C-lattice unconditional*. A sequence $\mathcal{X}$ is a Schauder basis for $\mathbb{X}$ if and only if it is a complete minimal system for which the *partial sum projections*

$$S_{\{n\in\mathbb{N}\colon n\le m\}}, \quad m \in \mathbb{N},$$

are uniformly bounded; besides, $\mathcal{F}(f) = \alpha(f)$ for all $f \in \mathbb{X}$.

Two sequences $(\boldsymbol{x}_n)_{n=1}^{\infty}$ and $(\boldsymbol{y}_n)_{n=1}^{\infty}$ in $\mathbb{X}$ are said to be *equivalent* if there is an isomorphism $T\colon[\mathcal{X}] \to [\mathcal{Y}]$ such that $T(\boldsymbol{x}_n) = \boldsymbol{y}_n$ for all $n \in \mathbb{N}$. If $\max\left\{\|T\|, \left\|T^{-1}\right\|\right\} \le C$ for some $C \ge 1$ we say that $\mathcal{X}$ and $\mathcal{Y}$ are C-equivalent. A *symmetric basis* will be a Schauder basis equivalent to all its permutations, and a *subsymmetric basis* will be an unconditional basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$ equivalent to $(\boldsymbol{x}_{\varphi(n)})_{n=1}^{\infty}$ for all increasing maps $\varphi\colon\mathbb{N} \to \mathbb{N}$. If $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$ is a symmetric (resp., subsymmetric) basis then there is a constant C such that for all one-to-one (resp., increasing) maps $\varphi\colon\mathbb{N} \to \mathbb{N}$ we have

(S) $\mathcal{X}$ is C-equivalent to $(\boldsymbol{x}_{\varphi(n)})_{n=1}^{\infty}$.

Symmetric bases are unconditional, hence subsymmetric. If a sequence $\mathcal{X}$ is C-lattice unconditional and (S) holds whenever $\varphi\colon\mathbb{N} \to \mathbb{N}$ is a one-to-one (resp., increasing) map, then we say that $\mathcal{X}$ is a C-symmetric (resp., C-subsymmetric) basis.

If the constant C appearing in the characterization of unconditionality, lattice unconditionality, symmetry, or subsymmetry is 1, we say that the corresponding property holds isometrically. Since these properties are linear, that is, their definitions only involve the boundedness of certain linear operators, any unconditional (resp., symmetric or subsymmetric) basis becomes isometrically lattice unconditional (resp., symmetric or subsymmetric) under a suitable renorming of the space (see [22] for the slightly more subtle case of subsymmetric bases).

A *symmetric sequence space* will be a quasi-Banach space $\mathbb{S} \subseteq \mathbb{F}^{\mathbb{N}}$ for which the unit vector system is an isometrically symmetric basis of its closed linear span in $\mathbb{S}$. In greedy approximation, several nonlinear forms of symmetry naturally appear. We next introduce some terminology in order to properly define them. Put

$$\mathbb{E} = \{\lambda \in \mathbb{F}\colon |\lambda| = 1\}, \quad \text{and} \quad \mathbb{D} = \{\lambda \in \mathbb{F}\colon |\lambda| \le 1\}.$$

Let $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$ be a sequence in a space $\mathbb{X}$. Given $A \subseteq \mathbb{N}$ finite and $\varepsilon = (\varepsilon_n)_{n\in A} \in \mathbb{E}^A$, we set

$$\mathbb{1}_{\varepsilon,A} = \sum_{n\in A} \varepsilon_n \, \boldsymbol{x}_n, \quad \mathbb{1}_A = \sum_{n\in A} \boldsymbol{x}_n.$$

The sequence $\mathcal{X}$ is said to be *C-symmetric for largest coefficients* (SLC for short), $1 \le C < \infty$, if

$$\left\| \mathbb{1}_{\varepsilon,A} + \sum_{n\in E} a_n \, \boldsymbol{x}_n \right\| \le C \left\| \mathbb{1}_{\delta,B} + \sum_{n\in E} a_n \, \boldsymbol{x}_n \right\| \tag{2.2}$$

for all pairwise disjoint finite subsets A, B, and E of $\mathbb{N}$, all $\varepsilon \in \mathbb{E}^A$, all $\delta \in \mathbb{E}^B$, and all $(a_n)_{n\in E}$ in $\mathbb{D}$.

If (2.2) holds in the case when $E = \emptyset$, we say that the sequence $\mathcal{X}$ is C-*superdemocratic*. If (2.2) holds under the additional restriction that ε and δ are constant, we say that $\mathcal{X}$ is C-democratic. In all cases, if the constant C is irrelevant we drop it from the notation, and if the condition holds with $C = 1$, we say that it holds isometrically. Isometric symmetry for largest coefficients was originally named Property (A) in [21].

A sequence is superdemocratic if and only if it is democratic and there is a constant C such that

$$\left\| \mathbb{1}_{\varepsilon,A} \right\| \le C \left\| \mathbb{1}_{\varepsilon,B} \right\|, \quad A \subseteq B \subseteq \mathbb{N}, |B| < \infty, \; \varepsilon \in \mathbb{E}^B. \tag{2.3}$$

Sequences that satisfy (2.3) are called *unconditional for constant coefficients* (UCC for short).

Superdemocracy can be characterized in terms of the *upper democracy function*, also known as *fundamental function* and denoted $\boldsymbol{\varphi_u}$, and the *lower democracy function* $\boldsymbol{\varphi_l}$ of the sequence. Namely, if for $m \in \mathbb{N}$ we set

$$\boldsymbol{\varphi_u}(m) = \sup_{\substack{|A| \le m \\ \varepsilon \in \mathbb{E}^A}} \left\| \mathbb{1}_{\varepsilon,A} \right\|, \quad \boldsymbol{\varphi_l}(m) = \inf_{\substack{|A| \ge m \\ \varepsilon \in \mathbb{E}^A}} \left\| \mathbb{1}_{\varepsilon,A} \right\|,$$

then $\mathcal{X}$ is C-superdemocratic if and only if $\boldsymbol{\varphi_u}(m) \le C \boldsymbol{\varphi_l}(m)$ for all $m \in \mathbb{N}$.

Let $\boldsymbol{\varphi_u^*}$ denote the fundamental function of the dual system $\mathcal{X}^*$ of the complete minimal system $\mathcal{X}$. If

$$\boldsymbol{\varphi_u}(m) \boldsymbol{\varphi_l^*}(m) \le C m, \quad m \in \mathbb{N}, \tag{2.4}$$

for some constant C, then both $\mathcal{X}$ and $\mathcal{X}^*$ are C-super-democratic. We call C-*bidemocratic* those complete minimal systems that satisfy (2.4) for a given constant C.

Finally, we say that a Banach space is squeezed between the symmetric sequence spaces $\mathbb{S}_1$ and $\mathbb{S}_2$ via a complete minimal system $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ if the *series transform*

$$(a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty a_n \, \boldsymbol{x}_n$$

defines a bounded operator from $\mathbb{S}_1$ into $\mathbb{X}$ and the coefficient transform is a bounded operator from $\mathbb{X}$ into $\mathbb{S}_2$. A complete minimal system is said to be *squeeze symmetric* if it is squeezed between two symmetric sequence spaces that are close to each other in the sense that the fundamental functions of their unit vector systems are equivalent.

Bidemocratic complete minimal systems are squeeze symmetric. In turn, squeeze symmetry is stronger than symmetry for largest coefficients.

Any democratic sequence $(\boldsymbol{x}_n)_{n=1}^\infty$ is *semi-normalized*, that is,

$$\inf_n \|\boldsymbol{x}_n\| > 0, \quad \sup_n \|\boldsymbol{x}_n\| < \infty.$$

In the case when $\mathcal{X}$ is a Schauder basis, its associated biorthogonal system $(\boldsymbol{x}_n, \boldsymbol{x}_n^*)_{n=1}^\infty$ is bounded, i.e.,

$$\sup_n \|\boldsymbol{x}_n\| \left\| \boldsymbol{x}_n^* \right\| < \infty. \tag{2.5}$$

Since any minimal system becomes normalized under rescaling, and semi-normalization is preserved by equivalence, it is natural to assume the minimal systems we deal with to be semi-normalized. In turn, (2.5) is a feature of the complete minimal system $\mathcal{X}$ that is convenient to impose in order to implement the *thresholding greedy algorithm* (TGA for short) with respect to $\mathcal{X}$. Note that if $\mathcal{X}$ is a semi-normalized complete minimal system whose associated biorthogonal system $(\boldsymbol{x}_n, \boldsymbol{x}_n^*)_{n=1}^\infty$ is bounded, then

$$\sup_n \left\| \boldsymbol{x}_n^* \right\| < \infty.$$

Hence, the coefficient transform maps $\mathbb{X}$ into c_0. Consequently, for each $f \in \mathbb{X}$ and $m \in \mathbb{N}$ there is a (not necessarily unique) greedy set A of f with $|A| = m$. Let $A_m(f)$ denote the one set for which $\max A$ is minimal. The TGA is the sequence $(\mathcal{G}_m)_{m=1}^\infty$ of nonlinear operators given by

$$\mathcal{G}_m : \mathbb{X} \to \mathbb{X}, \quad f \mapsto S_{A_m(f)}.$$

Given $f \in \mathbb{X}$, an arbitrary small perturbation of f yields a signal $g \in \mathbb{X}$ for which the greedy sets of any cardinality are unique. This observation leads to the paradigm that any functional property relative to the map $f \mapsto A_m(f)$, $f \in \mathbb{X}$, yields a property of the map

$$f \mapsto \{S_A(f) : A \text{ greedy set of } f,\ |A| = m\}, \quad f \in \mathbb{X}.$$

For simplicity, throughout this paper we will use the term *basis* to refer to a semi-normalized complete minimal system whose associated biorthogonal system is bounded. A *basic sequence* will be a sequence that is a basis of its closed linear span.

3. Greedy bases from an isometric point of view

A basis $\mathcal{X}$ of a Banach space $\mathbb{X}$ is said to be *greedy* if the TGA relative to the basis provides optimal sparse approximations, that is, there is a constant $G \geq 1$ such that

$$\|f - S_A(f)\| \leq G \, \|f - g\| \,, \tag{3.1}$$

whenever g is a linear combination of m vectors from $\mathcal{X}$ and A is a greedy set of $f \in \mathbb{X}$ with $|A| = m$. If (3.1) holds for a certain constant $G \geq 1$, we say that $\mathcal{X}$ is G-greedy.

Konyagin and Temlyakov [46] proved that $\mathcal{X}$ is a greedy basis if and only if it is democratic and unconditional. Quantitatively, optimizing the techniques from [46] yields that G-greedy bases are G-unconditional and G-democratic, and that if $\mathcal{X}$ is C-lattice unconditional and D-democratic, then $\mathcal{X}$ is G-greedy, where $G = C(1 + D)$. This last estimate implies that the best we can say about isometrically democratic and isometrically lattice unconditional bases is just that they are 2-greedy, which is not optimal for studying isometrically greedy bases.

The motivation behind this section lies in the analysis of the optimality of the thresholding greedy algorithm relative to bases in Banach spaces. This optimality is reflected in the sharpness of the constants that appear in the definitions of the different types of greedy-like bases. What justifies studying the "isometric" case in general is the fact that various approximation algorithms converge trivially when some appropriate constant is 1.

The first movers in this direction were Albiac and Wojtaszczyk, who in [21] characterized isometrically greedy bases using symmetry for largest coefficients instead of democracy. In fact, any G-greedy basis is G-unconditional and G-SSL, and any C-unconditional D-symmetric for largest coefficients basis is CD-greedy (see [3, Remark 3.8]). In particular, a basis is isometrically greedy if and only if it is isometrically unconditional and has Property (A). These estimates opened the door to study the following general question.

Question 3.1. Given a greedy basis of a Banach space $\mathbb{X}$, does it become 1-greedy under a suitable renorming of the space?

Any Banach space with a greedy basis can be renormed so that the basis becomes isometrically unconditional while the SSL constant does not increase. Thus, Question 3.1 reduces to the problem of finding a renorming of $\mathbb{X}$ so that the basis becomes isometrically symmetric for largest coefficients.

Isometrically symmetric bases are isometrically greedy, hence Banach spaces with a symmetric basis can be renormed so that the basis becomes isometrically greedy. Despite the fact that there are isometrically subsymmetric bases which are not isometrically greedy (see examples in [21]), the answer to Question 3.1 seems to be positive for all the subsymmetric bases found in the literature. For instance, this is the case with Garling sequence spaces, a type of spaces modelled after an example of Garling from [42] (cf. [17]).

Problem 1. *Does any subsymmetric basis of a Banach space become* 1*-greedy under a suitable renorming of the space?*

There are isometrically greedy bases that are not subsymmetric (see [37, Theorem 6.9]). So, Question 3.1 also makes sense for greedy bases that are not subsymmetric. The Haar system in $L_p := L_p([0, 1])$, $p \in (1, 2) \cup (2, \infty)$, is probably the most important example of such bases (see [54]).

Problem 2. *Let $1 < p < \infty$, $p \neq 2$. Is there a renorming of L_p so that the L_p-normalized Haar system becomes* 1*-greedy?*

Notice that the Haar system in L_p for $1 < p < \infty$ is bidemocratic, and so are the subsymmetric bases of any Banach space. Thus answering Question 3.1 in the positive for *bidemocratic* greedy bases would also answer in the positive Problems 1 and 2. In this regard we mention the following approximation to a solution of Problem 2.

Theorem 3.2 (*[34, Proposition 1.1]*)**.** *Let $\mathcal{X}$ be a bidemocratic greedy basis of a Banach space $\mathbb{X}$ and let $C > 1$. Then there is a renorming of $\mathbb{X}$ so that $\mathcal{X}$ becomes C-greedy.*

A more specific question than Question 3.1 that still covers Problem 2 is whether it admits a positive answer for spaces with nontrivial type. In fact, the fundamental function $\boldsymbol{\varphi_u} = (s_m)_{m=1}^\infty$ of any superdemocratic basis of a Banach space with nontrivial type has the *upper regularity property* (URP for short), that is,

$$s_{rm} \leq \frac{r}{2} s_m \quad m \in \mathbb{N},$$

for some $r \in \mathbb{N}$ (see [33, Proof of Proposition 4.1]). Besides, greedy bases, or, more generally, squeeze symmetric bases whose fundamental function has the URP, are bidemocratic (see [13, Lemma 9.8 and Proposition 10.17(iii)]). In this regard, it is natural to wonder whether the existence of a lattice structure on $\mathbb{X}$ could aid in obtaining a positive answer to Question 3.1.

Problem 3. *Let $\mathcal{X}$ be a greedy basis of a superreflexive Banach lattice $\mathbb{X}$. Is there a renorming of $\mathbb{X}$ so that $\mathcal{X}$ becomes* 1*-greedy?*

The fundamental function of the Haar system in L_p, $1 < p < \infty$, grows as $(m^{1/p})_{m=1}^\infty$ [53], so we could also address Problem 2 by focussing on the study of greedy basis whose fundamental functions grow as $(m^\alpha)_{m=1}^\infty$ for some $0 < \alpha < 1$. We point out that the answer to Question 3.1 is negative for greedy bases other than the canonical basis of ℓ_1, whose fundamental function grows as $(m)_{m=1}^\infty$ (see [37, Corollary 5]).

4. Isometric almost greediness

A basis $\mathcal{X}$ of a Banach space $\mathbb{X}$ is said to be *almost greedy* if the TGA provides optimal approximations by means of coordinate projections, that is, there is a constant $1 \le G$ (G-almost greedy) such that

$$\|f - S_A(f)\| \le G\, \|f - S_B(f)\|$$

whenever A is a greedy set of f and $B \subseteq \mathbb{N}$ satisfies $|A| = |B|$. In this section we are concerned about finding the optimal almost greediness constant.

Question 4.1. Given an almost greedy basis $\mathcal{X}$ of a Banach space $\mathbb{X}$, is there a renorming of $\mathbb{X}$ so that $\mathcal{X}$ becomes isometrically almost greedy or, at least, C-almost greedy with C arbitrarily close to 1?

Dilworth et al. provided a characterization of almost greedy bases that runs parallel to that of greedy ones. To achieve that they used a weaker form of unconditionality (namely, quasi-greediness) which was introduced in [46]. We recall that basis $\mathcal{X}$ is said to be *C-quasi-greedy*, $C \ge 1$, if $\|g\| \le C\, \|f\|$ for all $f \in \mathbb{X}$ and all greedy projections g of f. Of course, $\mathcal{X}$ is quasi-greedy if and only if there is a (possibly different) constant C such that

$$\|f - g\| \le C\, \|f\| \tag{4.1}$$

for all $f \in \mathbb{X}$ and all greedy projections g of f. If (4.1) holds for some $C \ge 1$, we say that $\mathcal{X}$ is *C-suppression quasi-greedy*.

Theorem 4.2 (*[33]*Theorem 3.3*). *Let $\mathcal{X}$ be a basis of a Banach space. Then $\mathcal{X}$ is almost greedy if and only if it is democratic and quasi-greedy.*

Theorem 4.2 is very nice but does not give quantitative estimates that could aid to address Question 4.1. It is known (see [3]) that a G-almost greedy basis is G-symmetric for largest coefficients and G-suppression quasi-greedy; and conversely, D-symmetric for largest coefficient and C-suppression quasi-greedy bases are CD-almost greedy. Hence, the almost greedy constant is close to one if and only if both the SLC constant and the suppression quasi-greedy constant are.

Being almost greedy is a weaker condition than being greedy. Taking this into consideration we could expect that obtaining renormings that improve the almost greedy constant would be easier than improving the greedy constant by renorming the space. However, in order to tackle Question 4.1 we will have to face a new obstruction: since quasi-greediness is not a linear condition, we should develop techniques for improving the suppression quasi-greedy constant. Another general question arises.

Question 4.3. Given a quasi-greedy basis $\mathcal{X}$ of a Banach space $\mathbb{X}$, is there a renorming of $\mathbb{X}$ so that $\mathcal{X}$ becomes 1-suppression quasi-greedy or, at least, C-suppression quasi-greedy with C arbitrarily close to 1?

Since 1-quasi greedy bases are 1-suppression unconditional [1], there is no point in replacing the suppression quasi-greedy constant with the quasi-greedy constant in the isometric part of Question 4.3. In contrast, the following problem makes perfect sense and remained open for a long time until very recently, when it was solved in the positive while this paper was under revision (see [24]).

Problem 4. *Is there a Banach space with a conditional* 1*-suppression quasi-greedy basis?*

Going back to almost greedy bases, we raise the problem whether it is possible to obtain the almost greedy version of Theorem 3.2.

Problem 5. *Given a bidemocratic almost greedy basis $\mathcal{X}$ of a Banach space $\mathbb{X}$ and $C > 1$, is there a renorming of $\mathbb{X}$ with respect to which $\mathcal{X}$ is C-almost greedy?*

When addressing Question 4.3 in the isometric case we should take into account that a basis is 1-almost greedy if and only if it has Property (A) (see [3]), so the problem of improving the suppression unconditionality constant disappears in this case, and we must only take care of improving the SLC constant. Besides, once it is shown that isometric symmetry for largest coefficients implies quasi-greediness, we should clarify whether it also implies a stronger condition.

A recent construction from [14] showed the existence of a renorming of ℓ_1 with respect to which the unit vector system has Property (A) but it is not 1-unconditional. However, this result does not answer the main question concerning Property (A).

Problem 6. *Does Property (A) imply unconditionality?*

It is even unknown whether there is a constant C so that any basis with Property (A) is C-unconditional. Note that Property (A) implies 1-suppression quasi-greediness.

5. Squeezing spaces between Lorentz spaces

In this section we will need the dual property of the URP. We say that a sequence $(s_m)_{m=1}^\infty$ in $(0, \infty)$ has the *lower regularity property* (LRP for short) if there is an integer $r \geq 2$ such

$$2s_m \leq s_{rm}, \quad m \in \mathbb{N}.$$

Given $0 < q \leq \infty$ and a nondecreasing sequence $\boldsymbol{\sigma} = (s_m)_{m=1}^\infty$, we adopt the convention that $s_0 = 0$, and define the Lorentz sequence space

$$d_q(\boldsymbol{\sigma}) = \left\{ f \in c_0 : \|f\|_{q,\boldsymbol{w}} := \left(\sum_{n=1}^\infty (b_n\, s_n)^q \frac{s_n - s_{n-1}}{s_n} \right)^{1/q} < \infty \right\},$$

where $(b_n)_{n=1}^\infty$ is the nonincreasing rearrangement of $|f|$, with the usual modification if $q = \infty$. If $\boldsymbol{\sigma}$ is doubling, that is,

$$\sup_m \frac{s_m}{s_{\lceil m/2 \rceil}} < \infty$$

then $d_q(\boldsymbol{\sigma})$ is a quasi-Banach space. In fact, $d_q(\boldsymbol{\sigma})$ is a Banach space provided that $1 \leq q < \infty$. However, $d_\infty(\boldsymbol{\sigma})$ is a Banach space if and only if $\boldsymbol{\sigma}$ has the URP. Besides, these spaces are reflexive if and only $1 < q < \infty$ and $\boldsymbol{\sigma}$ has the URP, and superreflexive if and only if $1 < q < \infty$ and $\boldsymbol{\sigma}$ has both the URP and the LRP (see [23]).

For a fixed doubling sequence $\boldsymbol{\sigma}$, $(d_q(\boldsymbol{\sigma}))_{q>0}$ is an increasing family of symmetric sequence spaces whose fundamental function grows as $\boldsymbol{\sigma}$. If $\boldsymbol{\sigma} = (m^{1/p})_{m=1}^\infty$ for some $0 < p < \infty$, then $d_q(\boldsymbol{\sigma})$ is the classical Lorentz sequence space $\ell_{p,q}$.

Given a basis $(\boldsymbol{x}_n)_{n=1}^\infty$ with dual basis $(\boldsymbol{x}_n^*)_{n=1}^\infty$, for $f \in \mathbb{X}$ we put

$$\varepsilon(f) = (\operatorname{sign}(\boldsymbol{x}_n^*(f)))_{n=1}^\infty \in \mathbb{E}^{\mathbb{N}},$$

where $\operatorname{sign}(0) = 1$ and $\operatorname{sign}(\lambda) = \lambda / |\lambda|$ otherwise.

The proof of Theorem 4.2 heavily depends on showing first that if a basis $\mathcal{X}$ is quasi-greedy then the *restricted truncation operators*

$$\mathcal{R}_m \colon \mathbb{X} \to \mathbb{X}, \quad f \mapsto \min_{n \in A_m(f)} \left| \boldsymbol{x}_n^*(f) \right| \mathbb{1}_{\varepsilon(f), A_m(f)}, \quad m \in \mathbb{N},$$

are uniformly bounded. This condition, coined in [11] as *truncation quasi-greediness* or TQG for short, is equivalent to the existence of a constant $C \ge 1$ so that

$$\min_{n \in A} \left| \boldsymbol{x}_n^*(f) \right| \left\| \mathbb{1}_{\varepsilon(f), A} \right\| \le C \left\| f \right\|$$

for all $f \in \mathbb{X}$ and all greedy sets A of f.

A basis $\mathcal{X}$ is truncation quasi-greedy if and only if the coefficient transform maps $\mathbb{X}$ into $d_\infty(\boldsymbol{\varphi_l})$. Moreover, for any basis $\mathcal{X}$ the series transform defines a bounded operator from $d_1(\boldsymbol{\varphi_u})$ into $\mathbb{X}$ (see [13, Section 9]). These results yield that a basis is squeeze symmetric if and only if it is truncation quasi-greedy and democratic; consequently, almost greedy bases are squeeze symmetric (cf. [2]). We also infer that if $\mathcal{X}$ is squeeze symmetric then $\mathbb{X}$ is squeezed between $d_1(\boldsymbol{\varphi_u})$ and $d_\infty(\boldsymbol{\varphi_u})$ via $\mathcal{X}$.

While $d_1(\boldsymbol{\varphi_u})$ is a Banach space, $d_\infty(\boldsymbol{\varphi_u})$ could be nonlocally convex. In fact, if $\boldsymbol{\varphi_u}(m) \approx m$ and the coefficient transform maps $\mathbb{X}$ into a symmetric Banach space then the basis $\mathcal{X}$ is equivalent to the standard unit vector basis of ℓ_1. To ensure that an almost greedy basis can be squeezed between two Banach spaces we must impose additional conditions, such as $\mathbb{X}$ having nontrivial type so that $\boldsymbol{\varphi_u}$ verifies the URP.

When we want to sandwich a Banach space $\mathbb{X}$ between two symmetric spaces $\mathbb{S}_1$, $\mathbb{S}_2$ that witness that the basis of $\mathbb{X}$ is squeeze-symmetric, in general we cannot guarantee that $\mathbb{S}_1$ and $\mathbb{S}_2$ retain all the features of $\mathbb{X}$. It may happen that $\mathbb{X}$ is locally convex, for instance, but that $\mathbb{S}_2$ is not. Or that $\mathbb{X}$ has nontrivial type but we lose that feature in one of the squeezing spaces. When dealing with superreflexive spaces this inconvenience disappears. In fact, if $\mathcal{X}$ is a quasi-greedy basis of a superreflexive Banach space then there is $1 < r < \infty$ such that the series transform defines a bounded operator from $d_r(\boldsymbol{\varphi_u})$ into $\mathbb{X}$ (see [23]). Besides, if the basis is almost greedy, a duality argument yields that the coefficient transform is a bounded operator from $\mathbb{X}$ into $d_q(\boldsymbol{\varphi_u})$ for some $1 < q < \infty$, whence $\mathbb{X}$ is squeezed between the superreflexive spaces $d_r(\boldsymbol{\varphi_u})$ and $d_q(\boldsymbol{\varphi_u})$. We wonder whether superreflexive spaces with a non-democratic quasi-greedy basis can be squeezed following a similar pattern.

Problem 7. *Let $\mathcal{X}$ be a quasi-greedy basis of a superreflexive Banach space $\mathbb{X}$. Is there $q < \infty$ such that the coefficient transform is a bounded operator from $\mathbb{X}$ into $d_q(\boldsymbol{\varphi_l})$?*

Here we point out that the answer to Problem 7 is positive for semi-normalized unconditional bases (see [4, Theorem 7.3]).

6. The TGA and Elton near unconditionality

A long standing question in basis theory, which was solved in the negative by Gowers and Maurey [43], asked whether all Banach spaces contained an unconditional basic sequence. Bearing in mind Rosenthal's theorem [49], which states that any bounded sequence in a Banach space either is equivalent to the canonical ℓ_1-basis or has a weakly Cauchy subsequence, the most natural way to look for a positive answer to this question was proving that any semi-normalized weakly null sequence has an unconditional basic sequence. When Maurey and Rosenthal [47] solved in the negative this question, the problem turned to finding subsequences of weakly null sequence that satisfy weaker forms of unconditionality.

In this ambience, Elton [41] introduced near unconditional bases and proved that any normalized weakly null sequence of a Banach space contains a nearly unconditionality subsequence. Suppose $\mathcal{X}$ is a basis of a Banach space $\mathbb{X}$. Set

$$\mathcal{Q} = \{f \in \mathbb{X} \colon \|f\|_\infty := \sup_n \left|\boldsymbol{x}_n^*(f)\right| \le 1\}.$$

Given a number $a \ge 0$ and $f \in \mathbb{X}$ put

$$A(a, f) := \{n \in \mathbb{N} \colon \left|\boldsymbol{x}_n^*(f)\right| \ge a\}.$$

The basis $\mathcal{X}$ is said to be *nearly unconditional* (NU for short) if for each $a \in (0, 1]$ there is a constant C such that

$$\|S_A(f)\| \le C\,\|f\|, \quad f \in \mathcal{Q},\ A \subseteq A(a, f). \tag{6.1}$$

The *unconditionality threshold function*

$$\phi \colon (0, 1] \to [1, \infty)$$

is defined for each a as the smallest value of the constant C in (6.1). Since a basis is unconditional if and only if ϕ is bounded, the unconditionality threshold function can be used to measure how far a basis is from being unconditional. We can also measure this distance by means of the *unconditionality parameters*

$$\boldsymbol{k}_m := \sup_{|A|\le m} \|S_A\|, \quad m \in \mathbb{N}.$$

The latter way of measuring unconditionality is coarser than the former. In fact,

$$\boldsymbol{k}_m \le 2 \sup_n \|\boldsymbol{x}_n\| \sup_n \left\|\boldsymbol{x}_n^*\right\| \phi(1/m), \quad m \in \mathbb{N},$$

(see [9, Lemma 6.1]).

Similarly to greedy, almost greedy, and squeeze symmetric bases, there is an unconditionality-like condition which combined with democracy characterizes symmetry for largest coefficients. This condition is called *quasi-greediness for largest coefficients*, or QGLC for short. We say that a basis $(\boldsymbol{x}_n)_{n=1}^\infty$ is C-QGLC if

$$\left\|\mathbb{1}_{\varepsilon,A}\right\| \le C \left\|\mathbb{1}_{\varepsilon,A} + f\right\|$$

for all $A \subseteq \mathbb{N}$ finite, all $\varepsilon \in \mathbb{E}^A$, and all $f \in \mathbb{X}$ with $\|\mathcal{F}(f)\|_\infty \le 1$ such that $\operatorname{supp}(f) \cap A = \emptyset$.

Oddly enough, quasi-greediness for largest coefficients and near unconditionality are the same property seen from different angles [8]. Besides, the unconditionality threshold function of a QGLC basis satisfies

$$\phi(a) = C a^{-\delta}, \quad 0 < a < 1, \tag{6.2}$$

for some $C \ge 1$ and some $\delta \in (0, \infty)$.

Truncation quasi-greedy bases of Banach spaces fulfil a better estimate, namely,

$$\phi(a) = C(1 - \log a), \quad 0 < a < 1, \tag{6.3}$$

for some constant C (see [9, Theorem 6.5]).

While QGLC bases need not be TQG [10], it seems to be unknown whether (6.2) is optimal for QGLC bases.

Problem 8. *Is there a nearly unconditional basis whose threshold unconditionality function does not have a logarithmic growth?*

Elton's aforementioned subsequence extraction principle has been improved. In fact, it is known that any semi-normalized weakly null sequence has a truncation quasi-greedy subsequence (see [5,36]). However, solving the quasi-greedy basic sequence problem has proven to be a more elusive task.

Problem 9. *Does any Banach space has a quasi-greedy basic sequence?*

Any Banach space with an unconditional spreading model which is nonequivalent to the canonical c_0-basis has a quasi-greedy basic sequence [36] (cf. [5]). So, in order to address Problem 9 it would suffice to focus on Banach spaces without an unconditional basis and whose unique unconditional spreading model is the standard c_0-basis.

7. Semi-greedy bases

A basis $\mathcal{X}$ of a Banach space $\mathbb{X}$ is said to be *semi-greedy* if there is a constant C such that for all $f \in \mathbb{X}$ and all greedy sets A of f there is $h \in [\boldsymbol{x}_n : n \in A]$ such that $\|f - h\| \le C \|f - g\|$ for all $g \in \mathbb{X}$ with $|\mathrm{supp}(g)| \le |A|$. This condition can be reformulated in terms of the Chebychev-type greedy algorithm, which assigns to every $f \in \mathbb{X}$ and every $m \in \mathbb{N}$ a vector $\mathcal{C}_m(f)$ that minimizes the distance $\|f - h\|$ when $h \in [\boldsymbol{x}_n : n \in A_m(f)]$.

Dilworth et al. [32] proved in 2003 that almost greedy bases are semi-greedy. Berná showed in 2019 that semi-greedy and almost greedy bases are equivalent concepts for Schauder bases [26], and a few years later, in 2023, Berasategui and Lassalle [25] demonstrated that this is true even when the hypothesis of being Schauder is dropped.

Semi-greedy bases, as well as all the types of bases we have considered so far, can be defined analogously in the wider framework of (not necessarily locally convex) quasi-Banach spaces. Most results on greedy-like bases that were originally stated and proved in Banach spaces hold for quasi-Banach spaces, although the constants involved could be worse. For instance, this is the case with the characterizations of greedy, almost greedy, squeeze symmetric, symmetric for largest coefficients, and super-democratic bases mentioned above (see [13]). However, it is unknown whether semi-greedy bases behave in the same way in non-locally convex quasi-Banach spaces.

Question 7.1. Are semi-greedy bases of quasi-Banach spaces almost greedy?

Fig. 1 represents the relations between the different forms of greediness and unconditionality we have considered. A double arrow means an implication between the two classes of bases involved. A double dashed arrow means that the implication holds under the extra assumption that the basis is democratic. A single dashed arrow means that the implication holds under the extra assumption that the space is locally convex.

8. Existence of greedy bases

There are well-known separable Banach spaces, such as L_1, without an unconditional basis, hence without a greedy basis.

If a Banach space has a (normalized) unconditional basis we find instances where this basis is, additionally, democratic thanks to the geometry of the space, hence that basis ends up being greedy. For example, this is what happens with subsymmetric bases, with the unit vector system of Tsirelson's space, or with the Haar system of both the dyadic Hardy space H_1 and L_p for $1 < p < \infty$.

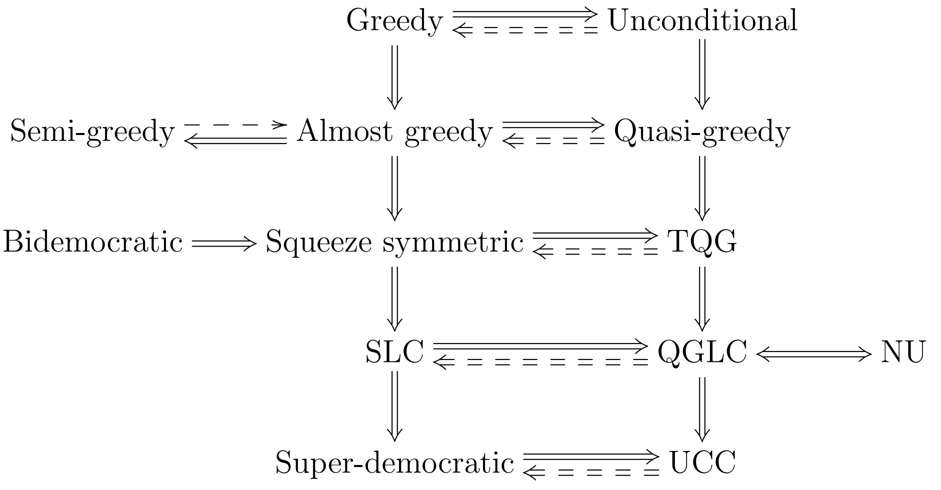

Fig. 1. Scale of greedy-like bases.

On the other hand, we find classical Banach spaces whose natural basis is unconditional and non-democratic, which might end up having a greedy basis or not. For instance, neither $\ell_p \oplus \ell_q$ nor the mixed-norm sequence spaces

$$Z_{p,q} := \ell_q(\ell_p),$$

$p, q \in [1, \infty]$, $p \neq q$, (if some index is ∞ we replace it for c_0) have a greedy basis [40,51]. However, for p and q in the same range on indices, the Besov spaces

$$B_{p,q} = \left(\oplus_{n=1}^{\infty} \ell_p^n\right)_{\ell_q}$$

have a greedy basis if and only if $1 < q < \infty$ [30]. The original Tsirelson's space does not even have a democratic basis [32]. Note that the space L_p for $p \in (1,2) \cup (2,\infty)$ has a greedy basis but its complemented subspace $\ell_p(\ell_2)$ does not!

Some problems remain open within this research topic. As far as sequence spaces are concerned, the more relevant problem seems to be settling the greedy basis structure of Nakano spaces.

Given $\boldsymbol{p} = (p_n)_{n=1}^{\infty}$ in $[1, \infty)$, the Nakano space (a.k.a. variable-exponent Lebesgue space) $\ell(\boldsymbol{p})$ consists of all sequences $(a_n)_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} |a_n|^{p_n} < \infty.$$

If $\sup_n p_n = \infty$ the space is not separable, in which case we replace it for its separable part.

The unit vector system is not a greedy basis for $\ell(\boldsymbol{p})$ unless $\ell(\boldsymbol{p}) = \ell_p$ for some p [15]. If the sequence $\boldsymbol{p}$ converges to 1 or to ∞, every complemented unconditional basis sequence of $\ell(\boldsymbol{p})$ is equivalent to a subbasis of the canonical basis [28], and we can infer that $\ell(\boldsymbol{p})$ does not have a greedy unless $\ell(\boldsymbol{p}) = \ell_1$ or $\ell(\boldsymbol{p}) = c_0$.

Problem 10. *Let* $\boldsymbol{p} = (p_n)_{n=1}^{\infty}$ *be a sequence in* $[1, \infty)$ *with* $\lim_n p_n \in (1, \infty)$. *Does* $\ell(\boldsymbol{p})$ *have a greedy basis?*

If $\lim_n p_n = 2$ it is even unknown whether $\ell(\boldsymbol{p})$ has a unique unconditional basis (up to rescaling, equivalence and permutation).

As for function spaces, it is known [65] that if $\mathbb{X}$ is a rearrangement invariant Banach space over $[0, 1]$, then the Haar system is a greedy basis for $\mathbb{X}$ if and only if $\mathbb{X} = L_p$ for some $1 < p < \infty$. The question of whether rearrangement invariant Banach spaces other that Lebesgue spaces have a greedy basis naturally arises. Here, the proximity of the space to L_1 or L_∞ may play a role, so it may be convenient to assume $\mathbb{X}$ to be superreflexive. We propose starting to explore this question in the particular case of Orlicz spaces.

Problem 11. *Let $F : [0, 1] \to [0, \infty)$ be a q-convex and r-concave Orlicz function for some $q > 1$ and $r < \infty$. Does ℓ_F have a greedy basis?*

The linear structure of nonlocally convex quasi-Banach spaces is more rigid than the structure of Banach spaces. It is plausible that any result stating the nonexistence of greedy basis for the spaces of a family of Banach spaces can be extended to their nonlocally convex relatives (see [13, Section 11.3]). In contrast, extending to the nonlocally convex setting an existence result may require a more careful analysis. Since certain locally convex Besov spaces have a greedy basis, we wonder about the existence of greedy bases for nonlocally convex Besov spaces.

Problem 12. *Let $0 < p < 1 < q < \infty$. Does $B_{p,q}$ have a greedy basis?*

If $p, q \in (0, 1]$, the spaces $B_{p,q}$ and $Z_{p,q}$ have a unique unconditional basis [20,45] whereas $B_{1,2}$ and $Z_{1,2}$ do not [27]. As of today it is unknown whether the spaces $B_{p,2}$ or $Z_{p,2}$ have a unique unconditional basis. Answering in the positive Problem 12 would yield a normalized unconditional basis $\mathcal{B}$ of $B_{p,2}$ nonequivalent to any permutation of the canonical basis. Since the unit vector systems of ℓ_p and ℓ_2 are the unique democratic subbases of the canonical basis $\mathcal{Z}$ of $Z_{p,2}$, $\mathcal{B}$ would not be equivalent to any subbasis of $\mathcal{Z}$. Therefore, the direct sum of $\mathcal{B}$ and $\mathcal{Z}$ would be a basis of a space isomorphic to $Z_{p,2}$ nonequivalent to any permutation of $\mathcal{Z}$.

9. Existence of almost greedy bases

Once we know that a certain Banach (or quasi-Banach) space does not have a greedy basis, we can ask ourselves whether it has, at least, an almost greedy basis. To tackle this problem, Dilworth et al. [32] invented a method for building conditional almost greedy bases that works for a broad class of spaces. To be precise, applying this method, which we call for short the DKK method, yields conditional almost greedy bases for any Banach space that contains a complemented copy of ℓ_1, and for any quasi-Banach space that contains a complemented copy of a superreflexive symmetric sequence space (see [7,16,32]). The DKK method produces almost greedy bases for $Z_{p,q}$ and $\ell_p \oplus \ell_q$ when $\max\{p, q\} > 1$ or $\min\{p, q\} \geq 1$. It also gives almost greedy bases for $B_{p,q}$ when $q > 1$ or $\min\{p, q\} \geq 1$. If $p, q \in (0, 1)$, then neither $\ell_p \oplus \ell_q$ nor $Z_{p,q}$ nor $B_{p,q}$ has an almost greedy basis unless $p = q$ [6]. The following problem raises the question on the existence of almost greedy bases in five of those spaces.

Problem 13. *Let $0 < p < 1$ and let $\mathbb{X}$ be $\ell_1 \oplus \ell_p$, $Z_{1,p}$, $Z_{p,1}$, $B_{1,p}$ or $B_{1,p}$. Does $\mathbb{X}$ have an almost greedy basis?*

We point out that apart from the DKK method, which does not seem to work to solve Problem 13, we do not find in the literature many other techniques that can be used for building almost greedy bases.

The first example of a conditional quasi-greedy basis, which turned out to be also democratic, was given by Konyagin and Temlyakov [46]. Although their technique allows small variations (see [13, Theorem 11.39]), it is far from being suited to solve Problem 13. Other constructions just give almost greedy bases for particular spaces. Let us mention a couple of results that yield bases for spaces to which the DKK method cannot be applied. The first one, which serves to construct a conditional almost greedy basis for ℓ_p, $0 < p \le 1$, asserts that the Lindenstrauss basis is conditional and almost greedy in ℓ_p (see [18,35]). The second one says that the Haar system is an almost greedy basic sequence of $\mathrm{BV}(\mathbb{R}^d)$, $d \ge 2$ [29,64].

10. A retrospective look at the role of Schauder bases for implementing the TGA

The TGA was initially studied for Schauder bases of Banach spaces. Wojtaszczyk took the lead in studying the TGA within the framework of complete minimal systems in quasi-Banach spaces, but his initiative did not have many followers. Since the natural order in the sequence of the positive integers does not play a significant role in implementing the greedy algorithm, it must be conceded that developing the theory for Schauder bases is somewhat unnatural and limiting. Besides, the fact of not taking for granted *a priori* Schauder's condition helps isolate the ingredients in the proofs of the important results. It could be argued that minimal systems that are not Schauder bases do not appear naturally, to the extent that the following important problem remains open.

Problem 14. *Is there a quasi-greedy basis that cannot be rearranged in such a way that it becomes a Schauder basis?*

If we replace quasi-greediness with truncation quasi-greediness in Problem 14, the answer to the question is positive. In fact, there are bidemocratic bases $(\boldsymbol{x}_n)_{n=1}^\infty$ such that $(\boldsymbol{x}_{\varphi(n)})_{n=1}^\infty$ is not a Schauder basis for any permutation φ of $\mathbb{N}$ (see [12]).

11. Dual bases of quasi-greedy bases

There are Banach spaces with a quasi-greedy basis whose dual basis is not quasi-greedy. Take, for instance, the Lindenstrauss basis of ℓ_1. In contrast, the dual basis of any bidemocratic quasi-greedy basis is quasi-greedy [33, Corollary 5.5]. Consequently, the dual basis of an almost greedy basis of a Banach space with nontrivial type is almost greedy. Since all known methods for building conditional quasi-greedy bases give almost greedy bases, and the dual basis of a normalized unconditional basis is obviously quasi-greedy, no example of a quasi-greedy basis of a Banach space with nontrivial type whose dual basis is not quasi-greedy is known. In particular, we pose the following problem.

Problem 15. *Let $1 < p < \infty$, $p \neq 2$. Does there exist a quasi-greedy basis of ℓ_p whose dual basis in not quasi-greedy?*

Note that any quasi-greedy basis of ℓ_2 is almost greedy [63, Theorem 3], whence its dual basis is quasi-greedy.

12. Banach envelopes of quasi-greedy bases

Given a quasi-Banach space $\mathbb{X}$ there is a pair $(\widehat{\mathbb{X}}, J_{\mathbb{X}})$ that satisfies the universal property associated with all pairs $(\mathbb{Y}, T)$ consisting of a Banach space $\mathbb{Y}$ and a linear contraction

$T:\mathbb{X} \to \mathbb{Y}$. We call $\widehat{\mathbb{X}}$ the *Banach envelope* of $\mathbb{X}$ and $J_{\mathbb{X}}:\mathbb{X} \to \widehat{\mathbb{X}}$ the *envelope map* of $\mathbb{X}$.

If $\mathcal{X}$ is a basis of $\mathbb{X}$, then $\widehat{\mathcal{X}} := J_{\mathbb{X}}(\mathcal{X})$ is a basis of $\widehat{\mathbb{X}}$ that inherits from $\mathcal{X}$ all its linear properties. For instance, if $\mathcal{X}$ is normalized and unconditional in $\mathbb{X}$ then $\widehat{\mathcal{X}}$ is (semi-normalized and) unconditional in $\widehat{\mathbb{X}}$.

As far as nonlinear properties are concerned, there are instances where $\mathcal{X}$ is greedy and $\widehat{\mathcal{X}}$ is not democratic (see [13, Section 11.7]), hence not greedy. Determining whether weaker, nonlinear forms of unconditionality pass to the Banach envelope is practically a wide open problem. Here we highlight the following important case.

Problem 16. *Let $\mathcal{X}$ be a quasi-greedy basis of a quasi-Banach space $\mathbb{X}$. Is $\widehat{\mathcal{X}}$ a quasi-greedy basis of $\widehat{\mathbb{X}}$?*

13. Weak Chebyshev greedy algorithm

So far we have concentrated and discussed a special case of sparse approximation with respect to a basis and a very specific algorithm to carry out such approximation, namely the Thresholding Greedy Algorithm. In our last section we mostly continue to discuss the case of bases but instead of the TGA we consider another greedy-type algorithm, which was introduced and studied for sparse approximation with respect to an arbitrary dictionary (see [55]).

Indeed, in many applications it is convenient to replace a basis by a more general system which may be redundant, that is to say, repetitions are allowed. This latter setting is much more complicated than the former (the basis case), however there is a solid justification of the importance of redundant systems in both theoretical questions and in practical applications in numerical analysis (see for instance [39,44,52]). The reader can find further discussion of this topic in the books [50,58,60,61] and the survey papers [56,57].

In a general setting we will be working in a Banach space $\mathbb{X}$ with a redundant system of elements that is called a dictionary $\mathcal{D}$. Recall that a set of elements (functions) $\mathcal{D}$ from $\mathbb{X}$ is a dictionary if each $g \in \mathcal{D}$ has $\|g\| = 1$, and the closure of $\mathcal{D}$ is $\mathbb{X}$.

A signal (or function) $f \in \mathbb{X}$ is said to be *m-sparse with respect to $\mathcal{D}$* if it admits a representation $f = \sum_{i=1}^{m} c_i\, \boldsymbol{g}_i$ with $\boldsymbol{g}_i \in \mathcal{D}$, $i = 1, \dots, m$. The set of all m-sparse elements is denoted by $\Sigma_m(\mathcal{D})$.

For a given function $f_0 \in \mathbb{X}$, the error of the best m-term approximation is defined as

$$\sigma_m(f_0, \mathcal{D}) := \inf_{g \in \Sigma_m(\mathcal{D})} \| f_0 - g \| .$$

In a broad sense, we are interested in the following fundamental question of sparse approximation with redundancy.

Question 13.1. How to design a practical algorithm relative to a dictionary that builds sparse approximations comparable (in the sense of error) to the best m-term approximations?

Of course, this is too big of a question which needs to be tackled in specific situations, but the pattern is the same in all of them, namely we introduce and study an approximation method given by a sequence of maps (an algorithm) $\mathcal{A} = (\mathcal{A}_m)_{m=1}^{\infty}$ relative to a dictionary $\mathcal{D}$ in a Banach space $\mathbb{X}$. That is, $\mathcal{A}_m(f)$ belongs to $\Sigma_m(\mathcal{D})$ for all $f \in \mathbb{X}$. Obviously, for any $f \in \mathbb{X}$ and any $m \in \mathbb{N}$ we have

$$\| f - A_m(f, \mathcal{D})\| \ge \sigma_m(f, \mathcal{D}).$$

We are interested in such pairs $(\mathcal{D}, \mathcal{A})$ for which the algorithm $\mathcal{A}$ provides approximation close to the best m-term approximation. In order to measure the efficiency of this algorithm we introduce the corresponding definitions.

Definition 13.2. We say that $\mathcal{D}$ is an *almost greedy dictionary* with respect to $\mathcal{A}$ if there exist constants C_1 and C_2 such that for any $f \in \mathbb{X}$ and $m \in \mathbb{N}$

$$\left\| f - \mathcal{A}_{\lceil C_1 m \rceil}(f, \mathcal{D}) \right\| \le C_2 \sigma_m(f, \mathcal{D}).$$

If $\mathcal{D}$ is an almost greedy dictionary with respect to $\mathcal{A}$ then $\mathcal{A}$ gives almost ideal sparse approximations; it provides a $\lceil C_1 m \rceil$-term approximant as good (up to a constant C_2) as the ideal m-term approximant for every $f \in \mathbb{X}$. In the case when $C_1 = 1$ we call $\mathcal{D}$ a *greedy dictionary*.

We also need a more general definition.

Definition 13.3. Let $\phi(u)$ be a function such that $\phi(u) \ge 1$. We say that $\mathcal{D}$ is a *ϕ-greedy dictionary with respect to the algorithm* $\mathcal{A}$ if there exists a constant C_3 such that for all $f \in \mathbb{X}$ and all $m \in \mathbb{N}$,

$$\left\| f - \mathcal{A}_{\lceil \phi(m) m \rceil}(f, \mathcal{D}) \right\| \le C_3 \sigma_m(f, \mathcal{D}).$$

It was shown in [59] that the Weak Chebyshev Greedy Algorithm, which we define momentarily, is a solution to Question 13.1 for a special class of dictionaries.

For a nonzero element $g \in \mathbb{X}$, by the Hahn–Banach theorem there exists a norming (or peak) functional for g, i.e., an element $F_g \in \mathbb{X}^*$ with $\left\| F_g \right\|_{\mathbb{X}^*} = 1$ and such that $F_g(g) = \|g\|_{\mathbb{X}}$. Let $f_0 \in \mathbb{X}$ be given. Then for each $m \ge 1$ and any *weakness parameter* $t \in (0, 1]$ we have the following inductive definition.

- Choose any element $\varphi_m := \varphi_m^{c,t} \in \mathcal{D}$ satisfying

$$\left| F_{f_{m-1}}(\varphi_m) \right| \ge t \sup_{g \in \mathcal{D}} \left| F_{f_{m-1}}(g) \right|.$$

- Put $\Phi_m := \Phi_m^t := \operatorname{span}(\varphi_j : 1 \le j \le m)$ and then define $G_m := G_m^{c,t}$ to be the best approximant to f_0 from Φ_m.
- Finally, let $f_m := f_m^{c,t} := f_0 - G_m$.

The Weak Chebyshev Greedy Algorithm (WCGA for short) (see [55]) is a generalization for Banach spaces of the Weak Orthogonal Matching Pursuit (WOMP). In a Hilbert space the WCGA coincides with the WOMP. The WOPM is very popular in signal processing, in particular in compressed sensing. In the case when $t = 1$, the WOMP is called Orthogonal Matching Pursuit (OMP).

We note that the properties of a given basis in a Banach space with respect to the TGA and with respect to the WCGA could be very different, both from a qualitative and a quantitative approach. To illustrate the differences in the performance of the TGA and WCGA we use the d-variate trigonometric system $\mathcal{T}^d$ as an example.

Proposition 13.4 (*[60, Theorem 2.2.1]*)**.** *Let $1 \le p \le \infty$, and set $h(p) = |1/2 - 1/p|$. There is a constant C such that*

$$\left\| \mathcal{G}_m(f, \mathcal{T}^d) \right\|_p \le C m^{h(p)} \|f\|_p, \quad m \in \mathbb{N},\ f \in L_p(\mathbb{T}^d).$$

Moreover, the extra factor $m^{h(p)}$ cannot be improved.

The following inequalities were obtained very recently (see [62]).

Proposition 13.5. *Let $2 \le p < \infty$. For any $f \in L_p$ and for each $m \in \mathbb{N}$,*

$$\left\|\mathcal{G}_m(f, \mathcal{T}^d)\right\|_p \le C(p) \, \|f\|_p^{2/p} \, \|f\|_{A_1(\mathcal{T}^d)}^{1-2/p}. \tag{13.1}$$

Let $1 \le p \le 2$. For any $f \in L_p$ and for each $m \in \mathbb{N}$,

$$\left\|\mathcal{G}_m(f, \mathcal{T}^d)\right\|_p \le C(p) \, \|f\|_p^{p/2} \, \|f\|_{A_1(\mathcal{T}^d)}^{1-p/2}, \tag{13.2}$$

where

$$\|f\|_{A_1(\mathcal{T}^d)} := \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \left|\hat{f}(\boldsymbol{k})\right|, \quad \hat{f}(\boldsymbol{k}) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(\boldsymbol{x}) e^{-i(\boldsymbol{k},\boldsymbol{x})} d\boldsymbol{x}.$$

Clearly, the $\|\cdot\|_{A_1(\mathcal{T}^d)}$ norm is stronger than the $\|\cdot\|_p$ norm. However, it is important that in (13.1) and (13.2) the extra factor $C(p)$ does not depend on m.

To compare those estimates with the ones we get for the WCGA we need to consider the real trigonometric system because the Weak Chebyshev Greedy Algorithm has been well studied mainly for real Banach spaces. The reader can find some results on the WCGA for complex Banach spaces in [31]. We will denote by $\mathcal{RT}$ the real trigonometric system

$$\{1, \sin(2\pi x), \cos(2\pi x), \dots, \sin(2\pi n x), \cos(2\pi n x), \dots\}$$

on $[0, 1]$ and will let $\mathcal{RT}_p$ be its normalized version in $L_p([0, 1])$. Let

$$\mathcal{RT}_p^d = \mathcal{RT}_p \times \cdots \times \mathcal{RT}_p$$

be the d-variate real trigonometric system. The following Lebesgue-type inequality for the WCGA was proved in [59].

Proposition 13.6. *Let $2 \le p < \infty$, $d \in \mathbb{N}$, and $t \in (0, 1]$. There are constants C_1 and C_2 such that the WCGA with weakness parameter t relative to the d-variate trigonometric system $\mathcal{D} = \mathcal{RT}_p^d$ in L_p gives*

$$\left\| f_{\lceil C_1 m \ln(m+1) \rceil} \right\|_p \le C_2 \sigma_m(f_0, \mathcal{D})_p, \quad m \in \mathbb{N}, \; f_0 \in L_p([0, 1]^d). \tag{13.3}$$

Problem 17 (*See [56, Problem 7.1]*). *Does (13.3) hold without the* $\ln(m+1)$ *factor?*

Proposition 13.6 is the first result on the Lebesgue-type inequalities for the WCGA with respect to the trigonometric system. It is a first step towards the solution of Problem 17, but the problem is still open.

The dissimilarities between the TGA and the WCGA can be realized as well from a qualitative point of view. Take for instance, the class of quasi-greedy bases relative to the TGA, which is a rather narrow subset of bases close in a certain sense to the set of unconditional bases. The situation is dramatically different for the WCGA. For example, if $\mathbb{X}$ is uniformly smooth then WCGA converges for each $f \in \mathbb{X}$ with respect to any dictionary in $\mathbb{X}$ (see [58, Ch. 6]).

There exists a general theory of the Lebesque-type inequalities for the WCGA with respect to dictionaries satisfying certain conditions. The reader can find some of the corresponding results in [59] and [61, Ch. 8]. We only present here some corollaries of those general results in the case of bases satisfying certain conditions. For $p \in (1, \infty)$ we use the notation $p' = p/(p-1)$.

Proposition 13.7. *Let $\mathcal{X}$ be a uniformly bounded orthogonal system normalized in $L_p(\Omega)$ for $1 < p < \infty$, where Ω is a bounded domain. Then, given $0 < t \le 1$ there are constants C_1 and C_2 depending on p, t and Ω, such that*

$$\left\| f_{\lceil C_1 m \ln(m+1)\rceil} \right\|_p \le C_2 \sigma_m(f_0, \mathcal{X})_p, \quad m \in \mathbb{N},\ f_0 \in L_p(\Omega),\ 2 \le p < \infty,$$

and

$$\left\| f_{\left\lceil C_1 m^{p'-1} \ln(m+1)\right\rceil} \right\|_p \le C_2 \sigma_m(f_0, \mathcal{X})_p, \quad m \in \mathbb{N},\ f_0 \in L_p(\Omega),\ 1 < p \le 2.$$

Proposition 13.8. *Let $2 \le p < \infty$, $d \in \mathbb{N}$, and $t \in (0, 1]$. Then the normalized d-variate Haar basis in L_p satisfies*

$$\left\| f_{\left\lceil C_1 m^{2/p'}\right\rceil} \right\|_p \le C_2 \sigma_m(f_0, \mathcal{H}_p^d)_p, \quad m \in \mathbb{N},\ f_0 \in L_p([0, 1]^d),$$

for some constants C_1 and C_2.

Proposition 13.9. *Let $1 < p \le 2$ and $t \in (0, 1]$. Then the univariate Haar basis in L_p satisfies*

$$\left\| f_{C_1 m} \right\|_p \le C_2 \sigma_m(f_0, \mathcal{H}_p)_p, \quad m \in \mathbb{N},\ f_0 \in L_p([0, 1]),$$

for some constants C_1 and C_2.

Proposition 13.10. *Let $\gamma \in (0, \infty)$, $1 < q \le 2$, and $t \in (0, 1]$. Let $\mathbb{X}$ be a superreflexive Banach space whose modulus of smoothness ρ satisfies $\rho(u) \le \gamma u^q$ for all $u > 0$. Assume that $\mathcal{X}$ is a normalized Schauder basis for $\mathbb{X}$. Then there are constants C_1 and C_2 such that*

$$\left\| f_{\left\lceil C_1 m^{q'} \ln(m+1)\right\rceil} \right\| \le C_2 \sigma_m(f_0, \mathcal{X}), \quad m \in \mathbb{N},\ f_0 \in \mathbb{X}.$$

Proposition 13.11. *Let $0 < t \le 1$ and $\mathcal{X}$ be a normalized quasi-greedy basis for L_p, $1 < p < \infty$. Then*

$$\left\| f_{\left\lceil C_1 m^{2(1-1/p)} \ln(m+1)\right\rceil} \right\|_p \le C_2 \sigma_m(f_0, \mathcal{X}), \quad m \in \mathbb{N},\ f_0 \in L_p,\ 2 \le p < \infty,$$

and

$$\left\| f_{\left\lceil C_1 m^{p'/2} \ln(m+1)\right\rceil} \right\|_p \le C_2 \sigma_m(f_0, \mathcal{X}), \quad m \in \mathbb{N},\ f_0 \in L_p,\ 1 < p \le 2,$$

for some constants C_1 and C_2.

Proposition 13.12. *Let $0 < t \le 1$ and $\mathcal{X}$ be a normalized uniformly bounded orthogonal quasi-greedy basis for L_p, $1 < p < \infty$ (for existence of such bases see [38,48]). Then*

$$\left\| f_{\lceil C_1\ m \ln(\ln(m+3))\rceil} \right\|_p \le C \sigma_m(f_0, \mathcal{X})_p, \quad m \in \mathbb{N},\ f_0 \in L_p,\ 2 \le p < \infty,$$

and

$$\left\| f_{\left\lceil C_1 m^{p'/2} \ln(\ln(m+3))\right\rceil} \right\|_p \le C_2 \sigma_m(f_0, \mathcal{X})_p, \quad m \in \mathbb{N},\ f_0 \in L_p,\ 1 < p \le 2,$$

for some constants C_1 and C_2.

The approximation method provided by WCGA is not as well studied as the TGA from the point of view of the Lebesgue-type inequalities and greedy-type bases. It is a very interesting, albeit very difficult, area of research. The reader can find recent results in this direction in the paper [31]. We close with some open problems of this aspect of the theory in the special case when we study sparse approximation with respect to bases instead of with respect to redundant dictionaries (see [61, p. 448]).

Problem 18. *Characterize almost greedy bases with respect to the WCGA for the Banach space L_p, $1 < p < \infty$.*

Problem 19. *Is the d-variate trigonometric system $\mathcal{RT}_p^d$ an almost greedy basis with respect to the WCGA in $L_p(\mathbb{T}^d)$, $1 < p < \infty$?*

Problem 20. *Is the univariate Haar basis $\mathcal{H}_p$ an almost greedy basis with respect to the WCGA in L_p, $2 < p < \infty$?*

Problem 21. *Is the d-variate Haar basis $\mathcal{H}_p^d$, $d \ge 2$, an almost greedy basis with respect to the WCGA in L_p, $1 < p < \infty$?*

Problem 22. *For each L_p, $1 < p < \infty$, find the best ϕ such that any Schauder basis is a ϕ-greedy basis with respect to the WCGA.*

Problem 23. *For each L_p, $1 < p < \infty$, find the best ϕ such that any unconditional basis is a ϕ-greedy basis with respect to the WCGA.*

Problem 24. *Is there a greedy-type algorithm $\mathcal{A}$ such that the multivariate Haar system $\mathcal{H}_p^d$ is an almost greedy basis of L_p, $1 < p < \infty$, with respect to $\mathcal{A}$?*

Problem 25. *In all Propositions 13.7–13.8, 13.10–13.12 we do not know the right ϕ which makes the corresponding bases ϕ-greedy with respect to the WCGA.*

Declaration of competing interest

No potential competing interest was reported by the authors.

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Data availability

No data was used for the research described in the article.

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