Some remarks on discretization

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Two settings

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- Functions belong to an N-dimensional subspace X_N. We call such results the Bernstein-type and Marcinkiewicz-type discretization theorems.
- Functions belong to a given function class.
  There are different settings and different ingredients, which play important role in this problem.

Sampling discretization with absolute error

Let $W \subset L_q(\Omega,\mu)$, $1 \leq q < \infty$, be a class of continuous on Ω functions. We are interested in estimating the following optimal errors of discretization of the L_q norm of functions from W

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$$er_m(W, L_q) := \inf_{\xi^1,\dots,\xi^m} \sup_{f \in W} \left| \|f\|_q^q - \frac{1}{m} \sum_{j=1}^{m} |f(\xi^j)|^q \right|,$$

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$$er_m^o(W,L_q) := \inf_{\xi^1,\dots,\xi^m;\lambda_1,\dots,\lambda_m} \sup_{f\in W} \left| \|f\|_q^q - \sum_{j=1}^m \lambda_j |f(\xi^j)|^q \right|.$$

Sampling discretization with relative error. Bernstein problem

Let W be the unit ball $X_N^q := \{f \in X_N : \|f\|_q \le 1\}$. Then in the case, say, $er_m(W, L_q) < \epsilon < 1$ we obtain for all $f \in X_N^q$

$$(1-\epsilon)\|f\|_q^q \le \frac{1}{m}\sum_{\nu=1}^m |f(\xi^\nu)|^q \le (1+\epsilon)\|f\|_q^q.$$

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Bernstein problem. In the case $q = \infty$ we define L_∞ as the space of continuous on Ω functions and ask for

$$C_1\|f\|_\infty \le \max_{1\le\nu\le m} |f(\xi^\nu)| \le \|f\|_\infty. \tag{1}$$

Marcinkiewicz problem

Let Ω be a compact subset of $\mathbb{R}^d$ with the probability measure μ. We say that a linear subspace X_N of the $L_q(\Omega)$, $1 \le q < \infty$, admits the Marcinkiewicz-type discretization theorem with parameters m, q and constants C_1, C_2 if there exists a set $\{\xi^\nu \in \Omega, \nu = 1, \dots, m\}$ such that for any $f \in X_N$ we have

$$C_1 \|f\|_q^q \le \frac{1}{m} \sum_{\nu=1}^{m} |f(\xi^\nu)|^q \le C_2 \|f\|_q^q. \qquad (2)$$

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We will also use a brief way to express the above property: $X_N \in \mathcal{M}(m, q)$ or $X_N \in \mathcal{M}(m, q, C_1, C_2)$.

Two important directions

- Universal Bernstein-type and Marcinkiewicz-type discretization theorems.

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- Constructive sets for good discretization.

Universal discretization problem

Let $\mathcal{X}_N := \{X_N^j\}_{j=1}^k$ be a collection of linear subspaces X_N^j of the $L_q(\Omega)$, $1 \le q \le \infty$. We say that a set $\{\xi^\nu \in \Omega, \nu = 1, \dots, m\}$ provides universal discretization for the collection $\mathcal{X}_N$ if, Let $\mathcal{X}_N := \{X_N^j\}_{j=1}^k$ be a collection of linear subspaces X_N^j of the $L_q(\Omega)$, $1 \le q \le \infty$. We say that a set $\{\xi^\nu \in \Omega, \nu = 1, \dots, m\}$ provides universal discretization for the collection $\mathcal{X}_N$ if, in the case $1 \le q < \infty$, there are two positive constants $C_i(d, q)$, $i = 1, 2$, such that for each $j \in [1, k]$ and any $f \in X_N^j$ we have

$$C_1(d, q)\|f\|_q^q \le \frac{1}{m}\sum_{\nu=1}^{m} |f(\xi^\nu)|^q \le C_2(d, q)\|f\|_q^q. \qquad (3)$$

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In the case $q = \infty$ for each $j \in [1,k]$ and any $f \in X_N^j$ we have

$$C_1(d)\|f\|_\infty \le \max_{1\le\nu\le m} |f(\xi^\nu)| \le \|f\|_\infty. \qquad (4)$$

A known result

We begin with the universal discretization for the collection of subspaces of trigonometric polynomials with frequencies from parallelepipeds (rectangles). For $\mathbf{s} \in \mathbb{Z}^d_+$ define

$$R(\mathbf{s}) := \{\mathbf{k} \in \mathbb{Z}^d : |k_j| < 2^{s_j}, \quad j = 1, \dots, d\}.$$

Clearly, $R(\mathbf{s}) = \Pi(\mathbf{N})$ with $N_j = 2^{s_j} - 1$. Consider the collection $\mathcal{C}(n,d) := \{\mathcal{T}(R(\mathbf{s})), \|\mathbf{s}\|_1 = n\}$.

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Theorem (1; VT, 2017)

For every $1 \leq q \leq \infty$ there exists a large enough constant $C(d, q)$, which depends only on d and q, such that for any $n \in \mathbb{N}$ there is a set $\xi(m) := \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$, with $m \leq C(d, q)2^n$ that provides universal discretization in L_q for the collection $\mathcal{C}(n, d)$.

Dispersion

Let $d \geq 2$ and $[0,1)^d$ be the d-dimensional unit cube. For $\mathbf{x}, \mathbf{y} \in [0,1)^d$ with $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{y} = (y_1, \dots, y_d)$ we write $\mathbf{x} < \mathbf{y}$ if this inequality holds coordinate-wise.

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$$\mathcal{B} := \{[\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in [0,1)^d, \mathbf{x} < \mathbf{y}\}.$$

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$$\mathcal{B} := \{[\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in [0,1)^d, \mathbf{x} < \mathbf{y}\}.$$

For $n \geq 1$ let T be a set of points in $[0,1)^d$ of cardinality $|T| = n$. The volume of the largest empty (from points of T) axis-parallel box, which can be inscribed in $[0,1)^d$, is called the dispersion of T:

$$\mathrm{disp}(T) := \sup_{B \in \mathcal{B} : B \cap T = \emptyset} vol(B).$$

Universal discretization in L_∞

Theorem (2; VT, 2017)

Let a set T with cardinality $|T| = 2^r =: m$ have dispersion satisfying the bound $disp(T) < C(d)2^{-r}$ with some constant $C(d)$. Then there exists a constant $c(d) \in \mathbb{N}$ such that the set $2\pi T := \{2\pi \mathbf{x} : \mathbf{x} \in T\}$ provides the universal discretization in L_∞ for the collection $\mathcal{C}(n,d)$ with $n = r - c(d)$.

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Theorem (3; VT, 2017)

Assume that $T \subset [0,1)^d$ is such that the set $2\pi T$ provides universal discretization in L_∞ for the collection $\mathcal{C}(n,d)$. Then there exists a positive constant $C(d)$ with the following property $disp(T) \le C(d)2^{-n}$.

Difinition of the (t, r, d)-net

Definition

A (t, r, d)-net (in base 2) is a set T of 2^r points in $[0, 1)^d$ such that each dyadic box
$[(a_1 - 1)2^{-s_1}, a_1 2^{-s_1}) \times \cdots \times [(a_d - 1)2^{-s_d}, a_d 2^{-s_d})$, $1 \le a_j \le 2^{s_j}$, $j = 1, \dots, d$, of volume 2^{t-r} contains exactly 2^t points of T.

Arbitrary trigonometric polynomials

For $n \in \mathbb{N}$ denote $\Pi_n := \Pi(\mathbf{N}) \cap \mathbb{Z}^d$ with $\mathbf{N} = (2^{n-1} - 1, \dots, 2^{n-1} - 1)$, where, as above, $\Pi(\mathbf{N}) := [-N_1, N_1] \times \cdots \times [-N_d, N_d]$. Then $|\Pi_n| = (2^n - 1)^d < 2^{dn}$. Let $v \in \mathbb{N}$ and $v \leq |\Pi_n|$. Consider

$$\mathcal{S}(v, n) := \{Q \subset \Pi_n : |Q| = v\}.$$

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$$\mathcal{S}(v, n) := \{Q \subset \Pi_n : |Q| = v\}.$$

Then it is easy to see that

$$|\mathcal{S}(v, n)| = \binom{|\Pi_n|}{v} < 2^{dnv}.$$

Universal discretization problem

We are interested in solving the following problem of universal discretization. For a given $\mathcal{S}(v, n)$ and $q \in [1, \infty)$ find a condition on m such that there exists a set $\xi = \{\xi^\nu\}_{\nu=1}^m$ with the property: for any $Q \in \mathcal{S}(v, n)$ and each $f \in \mathcal{T}(Q)$ we have

$$C_1(q,d)\|f\|_q^q \le \frac{1}{m}\sum_{\nu=1}^{m} |f(\xi^\nu)|^q \le C_2(q,d)\|f\|_q^q.$$

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We present results for $q=2$ and $q=1$.

The case $q = 2$

We begin with a general construction. Let $X_N = \text{span}(u_1, \dots, u_N)$, where $\{u_j\}_{j=1}^N$ is a real orthonormal system on $\mathbb{T}^d$. With each $\mathbf{x} \in \mathbb{T}^d$ we associate the matrix $G(\mathbf{x}) := [u_i(\mathbf{x})u_j(\mathbf{x})]_{i,j=1}^N$. Clearly, $G(\mathbf{x})$ is a symmetric matrix. For a set of points $\xi^k \in \mathbb{T}^d$, $k = 1, \dots, m$, and $f = \sum_{i=1}^N b_i u_i$ we have

$$\frac{1}{m}\sum_{k=1}^m f(\xi^k)^2 - \int_{\mathbb{T}^d} f(x)^2 d\mu = \mathbf{b}^T \left(\frac{1}{m}\sum_{k=1}^m G(\xi^k) - I \right) \mathbf{b},$$

where $\mathbf{b} = (b_1, \dots, b_N)^T$ is the column vector. Therefore,

$$\left| \frac{1}{m}\sum_{k=1}^m f(\xi^k)^2 - \int_{\mathbb{T}^d} f(x)^2 d\mu \right| \le \left\| \frac{1}{m}\sum_{k=1}^m G(\xi^k) - I \right\| \|\mathbf{b}\|_2^2.$$

Probability bound

We recall that the system $\{u_j\}_{j=1}^N$ satisfies Condition **E** if there exists a constant t such that

$$w(x) := \sum_{i=1}^{N} u_i(x)^2 \leq Nt^2.$$

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Let points $\mathbf{x}^k$, $k = 1, \dots, m$, be independent uniformly distributed on $\mathbb{T}^d$ random variables. Then with a help of deep results on random matrices it was proved that

$$\mathbb{P}\left\{\left\|\sum_{k=1}^{m}(G(\mathbf{x}^k) - I)\right\| \geq m\eta\right\} \leq N \exp\left(-\frac{m\eta^2}{ct^2N}\right)$$

with an absolute constant c.

The union bound

Consider real trigonometric polynomials from the collection $\mathcal{S}(v,n)$. Using the union bound for the probability we get that the probability of the event

$$\left\|\sum_{k=1}^{m}(G_Q(\mathbf{x}^k)-I)\right\| \le m\eta \quad \text{for all} \quad Q\in\mathcal{S}(v,n)$$

is bounded from below by

$$1-|\mathcal{S}(v,n)|v\exp\left(-\frac{m\eta^2}{cv}\right).$$

For any fixed $\eta\in(0,1/2]$ the above number is positive provided $m\ge C(d)\eta^{-2}v^2n$ with large enough $C(d)$. The above argument proves the following result.

Main result for $q = 2$

Theorem (Dai, Prymak, VT, Tikhonov, 2018.)

There exist three positive constants $C_i(d)$, $i = 1, 2, 3$, *such that for any* $n, v \in \mathbb{N}$ *and* $v \leq |\Pi_n|$ *there is a set* $\xi = \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$, *with* $m \leq C_1(d)v^2 n$, *which provides universal discretization in* L_2 *for the collection* $\mathcal{S}(v, n)$: *for any* $f \in \cup_{Q \in \mathcal{S}(v,n)} \mathcal{T}(Q)$

$$C_2(d)\|f\|_2^2 \leq \frac{1}{m}\sum_{\nu=1}^{m} |f(\xi^\nu)|^2 \leq C_3(d)\|f\|_2^2.$$

Case $q = 1$

Similar to the case $q = 2$ a result on the universal discretization for the collection $\mathcal{S}(v, n)$ will be derived from the probabilistic result on the Marcinkiewicz-type theorem for $\mathcal{T}(Q)$, $Q \subset \Pi_n$. However, the probabilistic technique used in the case of $q = 1$ is different from the probabilistic technique used in the case $q = 2$. The proof from VT, 2017, gives the following result.

Probability bound

Theorem (VT, 2017)

Let points $\mathbf{x}^j \in \mathbb{T}^d$, $j = 1, \dots, m$, *be independently and uniformly distributed on* $\mathbb{T}^d$. *There exist positive constants* $C_1(d)$, C_2, C_3, *and* $\kappa \in (0,1)$ *such that for any* $Q \subset \Pi_n$ *and* $m \ge yC_1(d)|Q|n^{7/2}$, $y \ge 1$,

$$\mathbb{P}\left\{\forall f \in \mathcal{T}(Q), \quad C_2\|f\|_1 \le \frac{1}{m}\sum_{j=1}^{m}|f(\mathbf{x}^j)| \le C_3\|f\|_1\right\} \ge 1 - \kappa^y.$$

The union bound

Therefore, using the union bound for probability we obtain the Marcinkiewicz-type inequalities for all $Q \in \mathcal{S}(v,n)$ with probability at least $1-|\mathcal{S}(v,n)|\kappa^y$. Choosing $y=y(v,n):=C(d)vn$ with large enough $C(d)$ we get

$$1-|\mathcal{S}(v,n)|\kappa^{y(v,n)}>0.$$

This argument implies the following result on universality in L_1.

Main result for $q = 1$

Theorem (Dai, Prymak, VT, Tikhonov, 2018.)

There exist three positive constants $C_1(d)$, C_2, C_3, *such that for any* $n, v \in \mathbb{N}$ *and* $v \le |\Pi_n|$ *there is a set* $\xi = \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$, *with* $m \le C_1(d) v^2 n^{9/2}$, *which provides universal discretization in* L_1 *for the collection* $\mathcal{S}(v,n)$*: for any* $f \in \cup_{Q \in \mathcal{S}(v,n)} \mathcal{T}(Q)$

$$C_2 \|f\|_1 \le \frac{1}{m} \sum_{\nu=1}^m |f(\xi^\nu)| \le C_3 \|f\|_1.$$

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- Equal weights.
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A probabilistic technique is used for the equal weights discretization. This helps us to obtain the corresponding theorems on the universal discretization.
We don't know how to use weighted discretization for the universal discretization.
Sampling discretization of the uniform norm (Bernstein-type problem) is very different from sampling discretization of the integral norms (Marcinkiewicz-type problem).