EAST JOURNAL ON APPROXIMATIONS
Volume 5, Number 3 (1999), 365-379

A REMARK ON GREEDY APPROXIMATION IN BANACH SPACES *

S. V. KONYAGIN

Dept. Mech. and Math., Moscow State University, Moscow 119899, RUSSIA

AND

V. N. TEMLYAKOV

Dept. Math., University of South Carolina, Columbia, SC 29208, USA

We introduce a concept of greedy basis for a Banach space. It is a basis Ψ such that for each element its m-term approximation with regard to Ψ can be realized (in the sense of order) by a greedy type algorithm. We prove that a basis is greedy basis if and only if it is unconditional and democratic. Democratic basis is a one such that the norm of any sum of its elements is determined (within the multiplication by two constants) by the number of summands. Some further discussion is also presented.

1. Introduction

Let a Banach space X with a basis $\Psi = \{\psi_k\}_{k=1}^{\infty}$, $\|\psi_k\| = 1$, $k = 1, 2, \ldots$, be given. We consider the following theoretical greedy algorithm. For a given element $f \in X$ we consider the expansion

$$f = \sum_{k=1}^{\infty} c_k(f)\psi_k.$$

*This research was supported by the National Science Foundation Grant DMS 9622925, by ONR Grant N00014-91-5-1076, and by DAAG55-98-10002

Let an element $f \in X$ be given. We call a permutation ρ, $\rho(j) = k_j$, $j = 1, 2, \ldots$, of the positive integers *decreasing* and write $\rho \in D(f)$ if

$$|c_{k_1}(f)| \geq |c_{k_2}(f)| \geq \ldots$$

In the case of strict inequalities here $D(f)$ consists of only one permutation. We define the m-th greedy approximant of f with regard to the basis Ψ corresponding to a permutation $\rho \in D(f)$ by the formula

$$G_m(f, \Psi, \rho) := \sum_{j=1}^{m} c_{k_j}(f)\psi_{k_j}.$$

This is a simple algorithm which describes a theoretical scheme (not computationally ready) for m-term approximation of an element f. In order to understand the efficiency of this algorithm we compare its accuracy with the best possible one when the approximant is chosen among all linear combinations of m terms from Ψ. We define the best m-term approximation with regard to Ψ as follows

$$\sigma_m(f, \Psi) := \inf_{c_k, \Lambda} \|f - \sum_{k \in \Lambda} c_k \psi_k\|,$$

where inf is taken over all coefficients c_k and sets of indices Λ with cardinality $\#\Lambda = m$. The best we can achieve with the algorithm G_m is

$$\|f - G_m(f, \Psi, \rho)\| = \sigma_m(f, \Psi),$$

or a little weaker

$$\|f - G_m(f, \Psi, \rho)\| \leq G\sigma_m(f, \Psi) \tag{1.1}$$

for all elements $f \in X$ with a constant $G = C(X, \Psi)$ independent of f and m.

Definition 1. We call a basis Ψ *greedy basis* if for every $f \in X$ there exists a permutation $\rho \in D(f)$ such that (1.1) holds.

Proposition 1. *If Ψ is a greedy basis, then* (1.1) *holds for any permutation $\rho \in D(f)$.*

Denote by $\mathcal{H}_p := \{H_k^p\}_{k=1}^{\infty}$ the Haar basis on $[0, 1)$ normalized in $L_p(0, 1)$: $H_1^p = 1$ on $[0, 1)$ and for $k = 2^n + l$, $n = 0, 1, \ldots$, $l = 1, 2, \ldots, 2^n$,

$$H_k^p = \begin{cases} 2^{n/p}, & x \in [(2l-2)2^{-n-1}, (2l-1)2^{-n-1}) \\ -2^{n/p}, & x \in [(2l-1)2^{-n-1}, 2l2^{-n-1}) \\ 0, & \text{otherwise.} \end{cases}$$

The following theorem (see [T1]) establishes existence of greedy bases for $L_p(0,1)$, $1 < p < \infty$.

Theorem A. *Let $1 < p < \infty$ and a basis Ψ be L_p-equivalent to the Haar basis $\mathcal{H}_p$. Then for any $f \in L_p(0,1)$ and any $\rho \in D(f)$ we have*

$$\|f - G_m(f, \Psi, \rho)\|_{L_p} \leq C(p, \Psi)\sigma_m(f, \Psi)_{L_p}$$

with a constant $C(p, \Psi)$ independent of f, ρ, and m.

We use in this theorem the following definition of the L_p-equivalence. We say that $\Psi = \{\psi_k\}_{k=1}^{\infty}$ is *L_p-equivalent* to $\mathcal{H} = \{H_k\}_{k=1}^{\infty}$ if for any finite set Λ and any coefficients c_k, $k \in \Lambda$, we have

$$C_1(p, \Psi)\|\sum_{k\in\Lambda} c_k H_k\|_{L_p} \leq \|\sum_{k\in\Lambda} c_k \psi_k\|_{L_p} \leq C_2(p, \Psi)\|\sum_{k\in\Lambda} c_k H_k\|_{L_p}$$

with two positive constants $C_1(p, \Psi)$, $C_2(p, \Psi)$ which may depend on p and Ψ.

Thus, each basis Ψ which is L_p-equivalent to the univariate Haar basis $\mathcal{H}_p$ is a greedy basis for $L_p(0,1)$, $1 < p < \infty$. We note that in the case of Hilbert space each orthonormal basis is a greedy basis with a constant $G = 1$ (see (1.1)).

We give now the definitions of unconditional and democratic bases.

Definition 2. A basis $\Psi = \{\psi_k\}_{k=1}^{\infty}$ of a Banach space X is said to be *unconditional* if for every choice of signs $\theta = \{\theta_k\}_{k=1}^{\infty}$, $\theta_k = 1$ or -1, $k = 1, 2, \ldots$, the linear operator M_θ, defined by

$$M_\theta\Big(\sum_{k=1}^{\infty} a_k\psi_k\Big) = \sum_{k=1}^{\infty} a_k\theta_k\psi_k,$$

is a bounded operator from X into X.

The uniform boundedness principle implies that the unconditional constant

$$K := K(X, \Psi) := \sup_\theta \|M_\theta\|$$

is finite.

Remark. There are several equivalent definitions of unconditional basis (see [LT], [KS]). For instance, instead of signs θ in Definition 2 one can take every choice of Boolean sequence $b = \{b_k\}_{k=1}^{\infty}$, $b_k = 0$, or 1, $k = 1, 2, \ldots$, and get an equivalent definition of unconditional basis.

The following theorem is a well-known fact about unconditional bases (see [LT], p.19).

Theorem B. *Let Ψ be an unconditional basis for X. Then for every choice of bounded scalars $\{\lambda_k\}_{k=1}^{\infty}$, we have*

$$\|\sum_{k=1}^{\infty} \lambda_k a_k \psi_k\| \le 2K \sup_k |\lambda_k| \|\sum_{k=1}^{\infty} a_k \psi_k\|$$

(in the case of real Banach space X we can take K instead of $2K$).

Definition 3. We say that a basis $\Psi = \{\psi_k\}_{k=1}^{\infty}$ is a *democratic* basis if for any two finite sets of indices P and Q with the same cardinality, $\#P = \#Q$, we have

$$\|\sum_{k \in P} \psi_k\| \le D \|\sum_{k \in Q} \psi_k\| \tag{1.2}$$

with a constant $D := D(X, \Psi)$ independent of P and Q.

We prove in Section 2 the following theorem.

Theorem 1. *A basis is greedy if and only if it is unconditional and democratic.*

In Section 3 we show that the democratic basis is not necessarily a unconditional and vice versa. This means that we need both conditions (unconditionality and democracy) in Theorem 1. Some other related concepts will be introduced and discussed in Section 3. In Section 4 we give examples of greedy bases for functional spaces.

2. Proof of Theorem 1

The proof of the direct part of the theorem, that any unconditional and democratic basis is a greedy basis, goes the same way as the proof of Theorem A (see [T1] and also [T2, Lemma 2.1]). For completeness we shall present this proof here. In fact we prove a little more, namely, that (1.1) holds for any $\rho \in D(f)$. This combined with Theorem 1 implies Proposition 1.

Take any $\epsilon > 0$ and find

$$p_m(f) := \sum_{k \in P} b_k \psi_k$$

such that $\#P = m$ and

$$\|f - p_m(f)\| \le \sigma_m(f, \Psi) + \epsilon. \tag{2.1}$$

For any finite set of indices Λ we denote by S_Λ the projector

$$S_\Lambda(f) := \sum_{k\in\Lambda} c_k(f)\psi_k.$$

The assumption that Ψ is unconditional implies that

$$\|f - S_P(f)\| \le K(\sigma_m(f,\Psi) + \epsilon). \tag{2.2}$$

Let $\rho \in D(f)$ and

$$G_m(f,\Psi,\rho) = \sum_{k\in Q} c_k(f)\psi_k = S_Q(f).$$

Then

$$\|f - G_m(f,\Psi,\rho)\| \le \|f - S_P(f)\| + \|S_P(f) - S_Q(f)\|. \tag{2.3}$$

The first term in the right-hand side of (2.3) is estimated in (2.2). We estimate now the second term. Clearly,

$$S_P(f) - S_Q(f) = S_{P\setminus Q}(f) - S_{Q\setminus P}(f). \tag{2.4}$$

Similarly to (2.2) we have

$$\|S_{Q\setminus P}(f)\| \le K(\sigma_m(f,\Psi) + \epsilon). \tag{2.5}$$

Let us estimate now $\|S_{P\setminus Q}(f)\|$. By the definition of greedy algorithm G_m we have

$$A := \max_{k\in P\setminus Q} |c_k(f)| \le \min_{k\in Q\setminus P} |c_k(f)| =: B. \tag{2.6}$$

Then, by the virtue of Theorem B we have

$$\|S_{P\setminus Q}(f)\| \le 2KA\|\sum_{k\in P\setminus Q} \psi_k\|, \tag{2.7}$$

and

$$\|S_{Q\setminus P}(f)\| \ge (2K)^{-1}B\|\sum_{k\in Q\setminus P} \psi_k\|. \tag{2.8}$$

By (1.2) we get

$$\|\sum_{k\in P\setminus Q} \psi_k\| \le D\|\sum_{k\in Q\setminus P} \psi_k\|. \tag{2.9}$$

Combining (2.7)–(2.9) we obtain

$$\|S_{P\setminus Q}(f)\| \le 4DK^2\|S_{Q\setminus P}(f)\|. \tag{2.10}$$

Using (2.5) and (2.10), we derive from (2.4) and (2.3) that

$$\|f - G_m(f, \Psi, \rho)\| \le 4DK^3(\sigma_m(f, \Psi) + \epsilon)$$

and, therefore, the inequality

$$\|f - G_m(f, \Psi, \rho)\| \le 4DK^3\sigma_m(f, \Psi)$$

holds.

We prove now the inverse part of the theorem, namely, that a greedy basis is always unconditional and democratic. Assume that a given basis Ψ satisfies (1.1) for all $f \in X$. We begin with the unconditionality. We shall prove that for each function $f \in X$ and any finite set Λ we have

$$\|S_\Lambda(f)\| \le (G+1)\|f\|, \tag{2.11}$$

where G is from (1.1). It is well-known (see Remark from the Introduction) that (2.11) implies that Ψ is a unconditional basis. Take a number N such that

$$N > \max_k |c_k(f)|$$

and consider a new function

$$g := f - S_\Lambda(f) + N\sum_{k\in\Lambda}\psi_k.$$

Then we obviously have

$$\sigma_m(g) \le \|f\|, \tag{2.12}$$

and

$$G_m(g) := G_m(g, \Psi, \rho) = N\sum_{k\in\Lambda}\psi_k. \tag{2.13}$$

Thus, by our assumption that Ψ is a greedy basis, we get

$$\|f - S_\Lambda(f)\| = \|g - G_m(g)\| \le G\sigma_m(g) \le G\|f\|.$$

This implies (2.11).

We proceed now to proving that Ψ is democratic. Let two finite sets P and Q, $\#P = \#Q = m$, be given. Take a third one Y such that $\#Y = m$ and $Y \cap P = \emptyset$, $Y \cap Q = \emptyset$. For a given finite set Λ denote

$$\psi_\Lambda := \sum_{k\in\Lambda} \psi_k.$$

Fix any $\epsilon > 0$ and consider the function

$$f := (1+\epsilon)\psi_Q + \psi_Y.$$

Then

$$\sigma_m(f) \le (1+\epsilon)\|\psi_Q\|$$

and

$$\|f - G_m(f)\| = \|\psi_Y\|.$$

Therefore, by the assumption that Ψ is greedy we get

$$\|\psi_Y\| \le G(1+\epsilon)\|\psi_Q\|. \tag{2.14}$$

Similarly,

$$\|\psi_P\| \le G(1+\epsilon)\|\psi_Y\|. \tag{2.15}$$

Finally, combining the above two inequalities and taking into account that ϵ is arbitrarily small, we obtain the estimate

$$\|\psi_P\| \le G^2\|\psi_Q\|.$$

This completes the proof of Theorem 1.

3. Examples

3.1. Unconditionality does not imply democracy

This follows from properties of the multivariate Haar system $\mathcal{H}_p^2 = \mathcal{H}_p \times \mathcal{H}_p$ defined as the tensor product of the univariate Haar systems $\mathcal{H}_p$. The system $\mathcal{H}_p^2$ is an unconditional basis for $L_p([0,1]^2)$, $1 < p < \infty$. However, there is an example (see [T2, Section 4]) suggested by R. Hochmuth that shows that $\mathcal{H}_p^2$ is not democratic for $p \ne 2$.

3.2. Democracy does not imply unconditionality

Let X be the set of all real sequences $x = (x_1, x_2, \dots)$ such that

$$\|x\|_X = \sup_{N \in \mathbf{N}} \left| \sum_{n=1}^{N} x_n \right|$$

is finite. Clearly, X equipped with the norm $\|\cdot\|_X$ is a Banach space. Let $\psi_k \in X$, $k = 1, 2, \dots$, be defined as

$$(\psi_k)_n = \begin{cases} 1, & n = k \\ 0, & n \neq k. \end{cases}$$

Denote by X_0 the subspace of X generated by the elements ψ_k. It is easy to see that $\{\psi_k\}$ is a democratic basis in X_0. However, it is not an unconditional basis since

$$\left\| \sum_{k=1}^{m} \psi_k \right\| = m, \quad \text{while} \quad \left\| \sum_{k=1}^{m} (-1)^k \psi_k \right\| = 1.$$

3.3. Superdemocracy does not imply unconditionality

It is clear that an unconditional and democratic basis Ψ satisfies the following inequality

$$\|\sum_{k \in P} \theta_k \psi_k\| \leq D_S \|\sum_{k \in Q} \epsilon_k \psi_k\| \tag{3.1}$$

for any two finite sets P and Q, $\#P = \#Q$, and any choices of signs $\theta_k = \pm 1$, $k \in P$, and $\epsilon_k = \pm 1$, $k \in Q$.

Definition 3.1. We say that a basis Ψ is a *superdemocratic* basis if it satisfies (3.1).

Theorem 1 implies that a greedy basis is a superdemocratic one. Now we will construct an example of superdemocratic basis which is not an unconditional basis and therefore, by Theorem 1, it is not a greedy basis.

Let X be the set of all real sequences $x = (x_1, x_2, \dots) \in l_2$ such that

$$\|x\|_1 = \sup_{N \in \mathbf{N}} \left| \sum_{n=1}^{N} x_n / \sqrt{n} \right|$$

is finite. Clearly, X equipped with the norm

$$\|\cdot\| = \max(\|\cdot\|_{l_2}, \|\cdot\|_1)$$

is a Banach space. Let $\psi_k \in X$, $k = 1, 2, \ldots$, be defined as

$$(\psi_k)_n = \begin{cases} 1, & n = k \\ 0, & n \neq k. \end{cases}$$

Denote by X_0 the subspace of X generated by the elements ψ_k. It is easy to see that $\Psi = \{\psi_k\}$ is a democratic basis in X_0. Moreover, it is superdemocratic: for any $k_1, \ldots, k_m$ and for any choice of signs,

$$\sqrt{m} \le \left\| \sum_{j=1}^{m} \pm \psi_{k_j} \right\| < 2\sqrt{m}. \tag{3.2}$$

Indeed, we have

$$\left\| \sum_{j=1}^{m} \pm \psi_{k_j} \right\|_{l_2} = \sqrt{m},$$

$$\left\| \sum_{j=1}^{m} \pm \psi_{k_j} \right\|_{1} \le \sum_{j=1}^{m} 1/\sqrt{j} < 2\sqrt{m},$$

and (3.2) follows. However, Ψ is not an unconditional basis since for $m \ge 2$

$$\left\| \sum_{k=1}^{m} \psi_k/\sqrt{k} \right\| \ge \sum_{k=1}^{m} 1/k \asymp \log m,$$

but

$$\left\| \sum_{k=1}^{m} (-1)^k \psi_k/\sqrt{k} \right\| \asymp \sqrt{\log m}.$$

3.4. A quasi-greedy basis is not necessarily an unconditional basis

It follows from the definition of greedy basis (see (1.1)) that the inequality

$$\|G_m(f, \Psi, \rho)\| \le (G+1)\|f\| \tag{3.3}$$

holds for all m and all $f \in X$, with some $\rho \in D(f)$.

Definition 3.2. We say that a basis Ψ is *quasi-greedy* if there exists a constant C_Q such that for any $f \in X$ and any finite set of indices Λ, having the property

$$\min_{k \in \Lambda} |c_k(f)| \ge \max_{k \notin \Lambda} |c_k(f)|, \tag{3.4}$$

we have

$$\|S_\Lambda(f,\Psi)\| = \|\sum_{k\in\Lambda} c_k(f)\psi_k\| \le C_Q\|f\|. \tag{3.5}$$

It is clear that for elements f, with a unique decreasing rearrangement of coefficients ($\#D(f) = 1$), the inequalities (3.3) and (3.5) are equivalent. Modifying slightly the coefficients and using the continuity argument we get that (3.3) and (3.5) are equivalent.

We shall prove now that the basis Ψ constructed in the previous subsection 3.3 is a quasi-greedy basis. Combining this with the result from 3.3 that Ψ is not unconditional we get the required statement of this subsection.

Assume $\|f\| = 1$. Then by the definition of $\|\cdot\|$ we have

$$\sum_{k=1}^{\infty} |c_k(f)|^2 \le 1 \tag{3.6}$$

and for any M,

$$|\sum_{k=1}^{M} c_k(f)k^{-1/2}| \le 1. \tag{3.7}$$

It is clear that for any Λ we have

$$\|S_\Lambda(f,\Psi)\|_{l_2} \le \|f\|_{l_2} \le 1. \tag{3.8}$$

We estimate now $\|S_\Lambda(f,\Psi)\|_1$. Let Λ be any set satisfying (3.4). Denote

$$\alpha := \min_{k\in\Lambda} |c_k(f)|.$$

If $\alpha = 0$, we get $S_\Lambda(f,\Psi) = f$ and (3.5) holds. Let $\alpha > 0$. For any N, set

$$\Lambda^+(N) := \{k\in\Lambda \quad : \quad k > N\}, \qquad \Lambda^-(N) := \{k\in\Lambda \quad : \quad k \le N\}.$$

We have

$$\begin{aligned} \sum_{k\in\Lambda^+(N)} |c_k(f)|k^{-1/2} &\le \Big(\sum_{k\in\Lambda^+(N)} |c_k(f)|^{3/2}\Big)^{2/3}\Big(\sum_{k>N} k^{-3/2}\Big)^{1/3} \\ &\ll N^{-1/6}\Big(\sum_{k\in\Lambda^+(N)} |c_k(f)|^{3/2}(|c_k(f)|/\alpha)^{1/2}\Big)^{2/3} \ll (\alpha^2 N)^{-1/6}. \end{aligned} \tag{3.9}$$

Choose $N_\alpha := [\alpha^{-2}] + 1$. Then for any $M \le N_\alpha$ we have by (3.7) that

$$\left|\sum_{k\in\Lambda^-(M)} c_k(f)k^{-1/2}\right| \le \left|\sum_{k=1}^{M} c_k(f)k^{-1/2}\right|$$

$$(3.10) \qquad + \left|\sum_{k\notin\Lambda^-(M),k\le M} c_k(f)k^{-1/2}\right|$$

$$\le 1+\alpha\sum_{k=1}^{M} k^{-1/2} \le 1+2\alpha M^{1/2} \ll 1.$$

For $M > N_\alpha$ we get using (3.9) and (3.10)

$$\left|\sum_{k\in\Lambda^-(M)} c_k(f)k^{-1/2}\right| \le \left|\sum_{k\in\Lambda^-(N_\alpha)} c_k(f)k^{-1/2}\right| + \sum_{k\in\Lambda^+(N_\alpha)} |c_k(f)|k^{-1/2} \ll 1.$$

Thus

$$\|S_\Lambda(f,\Psi)\|_1 \le C,$$

what completes the proof. □

The above example and Theorem 1 show that a quasi-greedy basis is not necessarily a greedy basis.

3.5. Some more relations

It follows directly from the definitions of unconditional basis and quasi-greedy basis that an unconditional basis is always a quasi-greedy basis. This and 3.1 show that quasi-greedy basis is not necessarily a democratic basis.

3.6. Symmetric bases

We say that two systems $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ of a Banach space X are X-equivalent if there are two positive constants C_1 and C_2 such that for any finite set Λ of indices and any numbers a_k, $k \in \Lambda$, we have

$$C_1\|\sum_{k\in\Lambda} a_k x_k\| \le \|\sum_{k\in\Lambda} a_k y_k\| \le C_2\|\sum_{k\in\Lambda} a_k x_k\|.$$

Definition 3.3. A basis $\{x_n\}_{n=1}^\infty$ of a Banach space X is said to be *symmetric* if, for any permutation ρ of the positive integers, $\{x_{\rho(n)}\}_{n=1}^\infty$ is X-equivalent to $\{x_n\}_{n=1}^\infty$.

Proposition 3.2. *A symmetric basis is a greedy basis.*

Proof. It is well-known ([LT]) that every symmetric basis is also unconditional. It follows directly from the definition that every symmetric basis is also democratic. Thus Theorem 1 implies Proposition 3.2. □

The following statement and Theorem A show that a greedy basis is not necessarily a symmetric basis. Thus the greedy bases are somewhere in between of unconditional bases and symmetric bases.

Proposition 3.3. *The univariate Haar basis $\mathcal{H}_p$ is not a symmetric basis for $L_p(0,1)$ if $p \neq 2$.*

Proof. This is known and follows for instance in the case of $1 < p < \infty$ from the estimate

$$\|\sum_{n=0}^{m} \sum_{k=2^n+1}^{2^{n+1}} 2^{-n/p} H_k^p\|_p \asymp m^{1/2}$$

which can be obtained by the Littlewood – Paley theorem (see [KT, Ch.3, S.3]) and from the simple observation that for any system $\{H_k^p\}_{k\in\Lambda}$ with disjoint supports we have

$$\|\sum_{k\in\Lambda} c_k H_k^p\|_p = (\sum_{k\in\Lambda} |c_k|^p)^{1/p}.$$

□

4. Applications to function spaces

4.1. Trigonometric subsystems

Consider first the case when $X \subset L_p(\mathbb{T})$ and Ψ is a subsystem of the trigonometric system. Let $\mathcal{N} := \{n_k\}_{k=1}^{\infty}$ be a sequence of different integers. Denote

$$L_p(\mathbb{T},\mathcal{N}) := \overline{\text{span}}(\{e^{in_kx}\}_{k=1}^{\infty})$$

where the closure is taken in L_p. We discuss the question: for which $\mathcal{N}$ the system $\mathcal{T}(\mathcal{N}) := \{e^{in_kx}, \ \ n_k \in \mathcal{N}\}$ is a greedy basis for $L_p(\mathbb{T},\mathcal{N})$? By Theorem 1 it should be an unconditional basis for $L_p(\mathbb{T},\mathcal{N})$. Thus, we get from Khinchin's inequality (see [KS, Ch.2, S.2]) that for

$$f(x) = \sum_{k=1}^{\infty} \hat{f}(n_k)e^{in_kx}$$

we have

$$\|f\|_{L_p} \asymp \Big(\sum_{k=1}^{\infty} |\hat{f}(n_k)|^2\Big)^{1/2} = \|f\|_{L_2}, \quad 1 \le p < \infty. \tag{4.1}$$

It is clear that (4.1) implies that $\mathcal{T}(\mathcal{N})$ is a democratic basis for $L_p(\mathbb{T},\mathcal{N})$. Thus, a basis $\mathcal{T}(\mathcal{N})$ is a greedy basis for $L_p(\mathbb{T},\mathcal{N})$ if and only if (4.1) holds for each $f \in L_p(\mathbb{T},\mathcal{N})$.

In the case of uniform norm ($p = \infty$) we get from Theorem 1 that $\mathcal{T}(\mathcal{N})$ should be an unconditional basis for $L_\infty(\mathbb{T},\mathcal{N})$. This implies in turn that we should have

$$\|f\|_\infty \asymp \sum_k |\hat{f}(n_k)| \quad \text{for all} \quad f \in L_\infty(\mathbb{T},\mathcal{N}). \tag{4.2}$$

Subsequences $\mathcal{N}$ with the property (4.2) are called the Sidon sets (see [K]). It is clear that (4.2) implies that $\mathcal{T}(\mathcal{N})$ is a greedy basis for $L_\infty(\mathbb{T},\mathcal{N})$. Therefore, $\mathcal{T}(\mathcal{N})$ is a greedy basis for $L_\infty(\mathbb{T},\mathcal{N})$ if and only if $\mathcal{N}$ is a Sidon's set.

4.2. Uniformly bounded systems

Let $\Psi := \{\psi_k(x)\}_{k=1}^\infty$ be a uniformly bounded on Ω system of real functions where Ω is a bounded domain in $\mathbb{R}^d$. Assume that $\|\psi_k\|_p := \|\psi_k\|_{L_p(\Omega)} = 1$, $k = 1, 2, \ldots$, for some $1 \le p < \infty$ and denote

$$X(\Psi, L_p) := \overline{\operatorname{span}}\{\psi_k\}_{k=1}^\infty,$$

where the closure is taken in $L_p(\Omega)$.

Proposition 4.1. *A system Ψ is a greedy basis for $X(\Psi, L_p)$ if and only if it is an unconditional basis for $X(\Psi, L_p)$.*

Proof. Theorem 1 implies that a greedy basis is always an unconditional basis. It remains to prove that a Ψ satisfying the above assumptions is a greedy basis. Theorem 1 shows that it is sufficient to prove that Ψ is a democratic basis. We remark first that our assumptions $\|\psi_k\|_p = 1$ and $\|\psi_k\|_\infty \le M$ imply that $\|\psi_k\|_2 \ge M_1 > 0$. For a function

$$f = \sum_k c_k(f)\psi_k$$

consider the square function

$$S(f) := \Big(\sum_k c_k(f)^2 \psi_k^2\Big)^{1/2}.$$

Then

$$\|S(f)\|_\infty \le M(\sum_k c_k(f)^2)^{1/2}, \tag{4.3}$$

and for all $1 \le p < \infty$,

$$\|S(f)\|_p \le |\Omega|^{1/p} M(\sum_k c_k(f)^2)^{1/2}. \tag{4.4}$$

Using the fact that Ψ is an unconditional basis we get by Khinchin's inequality (see [KS, Ch.2, S.2])

$$\|f\|_p \asymp \|S(f)\|_p. \tag{4.5}$$

We prove now that

$$\|S(f)\|_p \asymp (\sum_k c_k(f)^2)^{1/2}. \tag{4.6}$$

Indeed, the upper estimate follows from (4.3) and (4.4). The lower estimate in the case $p \ge 2$ follows from the inequalities

$$\|S(f)\|_p \gg \|S(f)\|_2 \gg (\sum_k c_k(f)^2)^{1/2}. \tag{4.7}$$

In the case $1 \le p < 2$ the lower estimate follows from (4.3), (4.7) and the inequality

$$\|S(f)\|_2^2 \le \|S(f)\|_\infty^{2-p} \|S(f)\|_p^p.$$

The relations (4.5) and (4.6) imply that Ψ is a democratic basis. □

Acknowledgments. The problem of characterizing greedy bases was formulated by Professor A. Pelczynski in the discussion of the second author's talk in the conference on Harmonic Analysis and Approximations (Armenia, September 18–25, 1998). Professor R. DeVore asked us about relations between quasi-greedy and greedy bases. Professor D. Kutzarova brought up to our attention the concept of symmetric bases. We are grateful to Professors R. DeVore, D. Kutzarova, and A. Pelczynski for their interest to this topic.

References

[K] J. P. KAHANE, Series de Fourier Absolument Convergentes, Springer-Verlag, Berlin, 1976.

[KS] B. S. KASHIN AND A. A. SAAKYAN, Orthogonal Series, American Math. Soc., Providence, R.I., 1989.

[LT] J. LINDENSTRAUSS AND L. TZAFRIRI, Classical Banach Spaces I, Springer-Verlag, Berlin, 1977.

[T1] V. N. TEMLYAKOV, The best m-term approximation and greedy algorithms, *Advances in Comp. Math.* **8** (1998), 249–265.

[T2] V. N. TEMLYAKOV, Non-linear m-term approximation with regard to the multivariate Haar system, *East J. Approx.* **4** (1998), 87–106.

Received May 13, 1999