Discretization and related questions

Spherical designs and other optimal configurations

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Definition

Let S^d be the unit sphere in $\mathbb{R}^{d+1}$ with normalized Lebesgue measure $d\mu_d$. A set of points $x_1, \dots, x_N \in S^d$ is called a *spherical t-design* if

$$\int_{S^d} P(x) d\mu_d(x) = \frac{1}{N} \sum_{i=1}^{N} P(x_i)$$

for all algebraic polynomials in $d+1$ variables and of total degree at most t.

Main question

What is the minimal number of points in a spherical t-design in S^d?

Motivation

Bernstein problem on equal weight quadrature:
What is the minimal number $N = N(t)$ such that for some fixed collection of points $x_1, \dots, x_N \in [-1, 1]$ the equation

$$\frac{1}{2}\int_{-1}^{1} P(x)dx = \frac{1}{N}\sum_{i=1}^{N} P(x_i)$$

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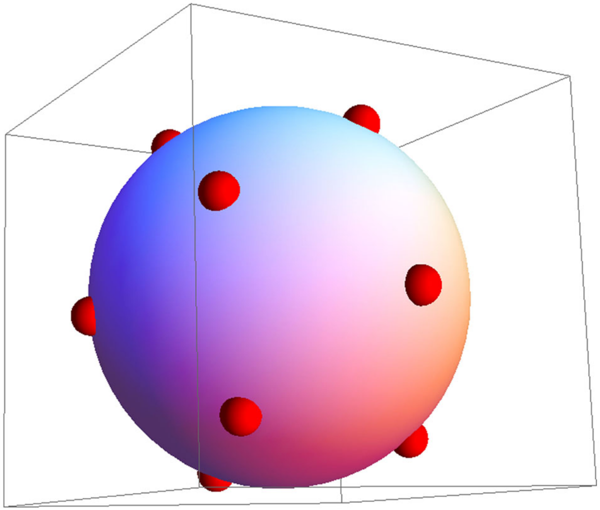
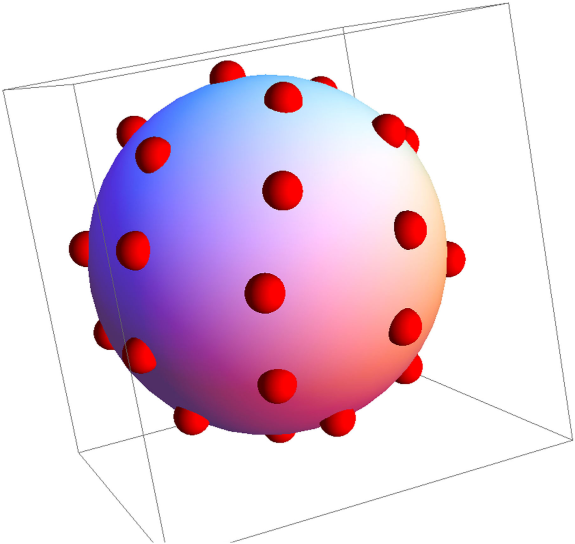
Answer: $N = O(t^2)$.
Claim: Projection of a spherical t-design in S^2 to any diameter is above mentioned quadrature.

5-design consisting of 12 points (icosahedron)



6-design consisting of 32 points



Lower bounds

For each $t \in \mathbb{N}$ denote by $N(d,t)$ the minimal number of points in a spherical t-design on S^d. The following lower bounds are proved by Delsarte, Goethals and Seidel in 1977:

$$N(d,t) \geq \binom{d+k}{d} + \binom{d+k-1}{d}, \ t = 2k,$$

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Corollary.

$$N(d,t) \geq c_d t^d.$$

Tight designs

Designs attaining these bounds are called *tight*.

Table of known tight designs.

dimension	$\sharp$ of points	strength	comment
1	t	t-1	t-gon
t	t+2	2	simplex
t	2t+2	3	octachedron
2	12	5	icosahedron
5	27	4	Shläfli
6	56	5	kissing
7	240	7	E_8 roots
21	275	4	kissing
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Tight designs with $d \geq 2$ may exist only for $t = 4, 5, 7$ or 11 (Bannai and Damerell).

Example

$X = \{(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}), (\pm 1, 0, 0, 0), \ldots, (0, 0, 0, \pm 1)\}$.
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There is 3-parameter family of 5-designs on S^3 consisting of 24 points.
(Cohn, Conway, Elkies, Kumar' 07)

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Conjecture. $N(d,t) \leq C_d t^d$.

Main result

We have proved the following

Theorem 1. (B., Radchenko, Viazovska) *For each $N \geq C_d t^d$ there exists a spherical t-design in S^d consisting of N points, where C_d is large enough.*

Idea of the proof

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Find a good starting configuration of N points on S^d which is "almost" a t-design.

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Find a good starting configuration of N points on S^d which is "almost" a t-design.

Step 2

Using topological degree theory prove that we can slightly move these points so that they become a t-design.

The space of polynomials

Let $\mathcal{P}_t$ be the vector space of polynomials P of degree $\leq t$ on S^d such that

$$\int_{S^d} P(x) d\mu_d(x) = 0.$$

We can define an inner product on $\mathcal{P}_t$ by

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For each point $x \in S^d$ there exists a unique polynomial $G_x \in \mathcal{P}_t$ such that

$$\langle G_x, Q \rangle = Q(x) \text{ for all } Q \in \mathcal{P}_t.$$

Then, the set of points $x_1, \ldots, x_N \in S^d$ forms a spherical design if and only if

$$G_{x_1} + \ldots + G_{x_N} = 0.$$

Area-regular partitions

Let $\mathcal{R} = \{R_1, \ldots, R_N\}$ be a finite collection of closed, non-overlapping (i.e., having no common interior points) regions $R_i \subset S^d$ such that $\cup_{i=1}^{N} R_i = S^d$.

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Theorem KS. (Kuijlaars, Saff ' 98)
For each $N \in \mathbb{N}$ there exists an area-regular partition $\mathcal{R} = \{R_1, \ldots, R_N\}$ such that $\|\mathcal{R}\| \leq c_d N^{-1/d}$ for some constant c_d.

Marcinkiewich-Zygmund inequality on the sphere

Theorem MNW. (Mhaskar, Narcowich, Ward '00) There exist constants r_d and N_d such that for each area-regular partition $\mathcal{R} = \{R_1, \ldots, R_N\}$ with $\|\mathcal{R}\| < \frac{r_d}{m}$, each collection of points $x_i \in R_i$, $i = \overline{1, \ldots, N}$ and each algebraic polynomial P of total degree $m > N_d$ the following inequality

$$\frac{1}{2} \int_{S^d} |P(x)| dx < \frac{1}{N} \sum_{i=1}^{N} |P(x_i)| < \frac{3}{2} \int_{S^d} |P(x)| dx$$

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holds.

Corollary.

$$\frac{1}{3\sqrt{d}}\int_{S^d} |\nabla P(x)|d\mu_d(x) \leq \frac{1}{N}\sum_{i=1}^{N} |\nabla P(x_i)| \leq 3\sqrt{d}\int_{S^d} |\nabla P(x)|d\mu_d(x).$$

Topological theorem

Theorem B. *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous mapping and* Ω *be an open bounded subset with the boundary* $\partial\Omega$ *such that* $0 \in \Omega \subset \mathbb{R}^n$. *If* $(x, f(x)) > 0$ *for all* $x \in \partial\Omega$, *then there exists* $x \in \Omega$ *satisfying* $f(x) = 0$.

The key lemma

Consider the following open subset of $\mathcal{P}_t$

$$\Omega := \left\{ P \in \mathcal{P}_t \,\middle|\, \int_{S^d} |\nabla P(x)| d\mu_d(x) < 1 \right\}.$$

Lemma If $N > C_d t^d$ then there are continuous mappings $x_i : \mathcal{P}_t \to S^d$ such that for all $P \in \partial\Omega$,

$$\frac{1}{N} \sum_{i=1}^{N} P(x_i(P)) > 0.$$

Proof of Theorem 1

Let $f : \mathcal{P}_t \to \mathcal{P}_t$ be defined by

$$f(P) := G_{x_1(P)} + \ldots + G_{x_N(P)}.$$

Clearly

$$(P, f(P)) = \sum_{i=1}^{N} P(x_i(P))$$

for each $P \in \mathcal{P}_t$.

Theorem B applied for the mapping f, the vector space $\mathcal{P}_t$, and the subset Ω gives us the existence of a polynomial $P \in \mathcal{P}_t$ such that $f(P) = 0$. Hence, the components of $F(P) = (x_1(P), \ldots, x_N(P))$ form a spherical t-design in S^d consisting of N points.

How to prove Lemma?

Lemma is “visible”.

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Lemma is "visible". To prove it we use a result on area-regular partitions (Kuijlaars, Saff) and the Marcinkiewicz-Zygmund inequality for the sphere (Mhaskar, Narcowich, and Ward)

Well-separated spherical designs

There exists well separated spherical t-designs in S^d of cardinality $O(t^d)$.

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Theorem 2. *For each $N \geq C_d t^d$ there exists a spherical t-design in S^d consisting of N points, such that $\mathrm{dist}(x_i, x_j) \geq \lambda_d N^{-1/d}$ for $i \neq j$, where C_d and λ_d depending only on d.*

Hardin-Sloane conjecture

Conjecture:

$$N(2,t) \le (\frac{1}{2} + o(1))t^2, \quad \text{as } t \to \infty.$$

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Motivation: $dim\, P_t = t^2 + 2t$. S^2 has dimension 2.

Other Generalizations

1. (D. Kane, 2015) $N(d,t) \leq dim(\mathcal{P}_{d,t})^2$.

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2. (Ujué Etayo, Jordi Marzo, Joaquim Ortega-Cerdà, 2018) Asymptotically optimal designs on compact algebraic manifolds

THANK YOU!