Sampling discretization of integral norms

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Discretization and related questions, March 9-12, 2021

Joints works:

[DPSTT1] F. Dai, A. Prymak,A. Shadrin, V.N. Temlyakov, S. Tikhonov (2020): Entropy numbers and Marcinkiewicz-type discretization theorem. arXiv:2001.10636.

[DPSTT2]: F. Dai, A. Prymak,A. Shadrin, V.N. Temlyakov, S. Tikhonov (2020): Sampling discretization of integral norms. arXiv:2001.09320.

[DP]: F. Dai, A. Prymak (2020): L_p-Bernstein inequalities on C^2-domains. arXiv:2010.06728.

Notations

- (Ω, μ): a probability space with probability measure μ.
- $L_p(\Omega)$: the Lebesgue L_p-space on (Ω, μ) with norm $\|\cdot\|_p$.
- X_N: an N-dimensional subspace of $L_p(\Omega)$ for some $1 \le p \le \infty$.
- WLOG: functions in X_N are defined everywhere on Ω. Indeed, we may first fix a basis $\{\varphi_1, \cdots, \varphi_N\}$ of X_N, and assume that each φ_i is defined EVERYWHERE on Ω.

Main purpose:

Discretize L_p-norms of $f \in X_N$ for $1 \leq p < \infty$:

$$C_1 \int_\Omega |f|^p \, d\mu \leq \sum_{\nu=1}^{m} \lambda_\nu |f(\xi^\nu)|^p \leq C_2 \int_\Omega |f|^p \, d\mu, \qquad (1)$$

where $\xi^j \in \Omega$, $\lambda_j > 0$, $1 \leq j \leq m$, both the ξ^j and the λ_j are independent of f, and $C_1, C_2 > 0$ are independent of f and N.

Of particular interest are the case when $\lambda_1 = \cdots = \lambda_m = \frac{1}{m}$, and the case when $C_1 = 1 - \varepsilon$ and $C_2 = 1 + \varepsilon$ for a given $\varepsilon \in (0, 1)$.

Organization of the talk

The talk consists of the following two parts:

- **Part I.** Discretization in a general setting, where X_N is an N-dimensional subspace of $L_p(\Omega)$ and (Ω, μ) is a probability space.

- **Part II.** Discretization in a specific setting, where
  - (a) $\Omega \subset \mathbb{R}^d$ is a compact domain with C^2-boundary;
  - (b) $d\mu = \frac{1}{|\Omega|} dx$(Lebesgue measure);
  - (c) $X_N = \Pi_n^d$ (the space of all algebraic polynomials in d variables of total degree $\leq n$).

Methods used in the two parts

- Part I uses a probabilistic approach, where we consider $\xi^1, \cdots, \xi^m \in \Omega$ as independent random pts, uniformly distributed in (Ω, μ). This approach is not constructive, but the resulting implicit constants are in general independent of the domain Ω.
- Part II uses the classical approach in polynomial approximation. Crucial tool: a new L^p-Bernstein-Markov inequality for algebraic polynomials on C^2 domains. This approach is constructive.

Part I: Discretization in the general setting

Most of the results in Part I can be found in:

[DPSTT1] F. Dai, A. Prymak,A. Shadrin, V.N. Temlyakov, S. Tikhonov (2020): Entropy numbers and Marcinkiewicz-type discretization theorem. arXiv:2001.10636.

[DPSTT2]: F. Dai, A. Prymak,A. Shadrin, V.N. Temlyakov, S. Tikhonov (2020): Sampling discretization of integral norms. arXiv:2001.09320.

Theorem A. (Rudelson 1999, Temlyakov 2018) Assume that $X_N \subset L_\infty$ satisfies

$$\sup_{f\in X_N} \frac{\|f\|_\infty}{\|f\|_2} \le (tN)^{\frac{1}{2}} \ \text{ for some } t > 0. \tag{2}$$

Then $\forall \epsilon \in (0,1)$, $\exists \{\xi^1, \cdots, \xi^m\} \subset \Omega$ s.t.

$$N \le m \le C\frac{t}{\epsilon^2} N \log N \ \text{ and}$$

$$(1-\epsilon)\|f\|_2^2 \le \frac{1}{m}\sum_{j=1}^m |f(\xi^j)|^2 \le (1+\epsilon)\|f\|_2^2, \ \ \forall f \in X_N.$$

Theorem B. (Limonova and Temlyakov 2020) Let $\Omega \subset \mathbb{R}^d$ be a compact subset equipped with a probability measure μ. Assume that $X_N \subset L_\infty(\Omega)$ satisfies

$$\sup_{f \in X_N} \frac{\|f\|_\infty}{\|f\|_2} \leq (tN)^{\frac{1}{2}} \quad \text{for some } t > 0. \tag{3}$$

Then $\exists \{\xi^1, \cdots, \xi^m\} \subset \Omega$ s.t.

$$N \leq m \leq C_3 tN \quad \text{and}$$

$$C_4 \|f\|_2 \leq \left(\frac{1}{m} \sum_{j=1}^{m} |f(x_j)|^2 \right)^{\frac{1}{2}} \leq C_5 \|f\|_2, \ \ \forall f \in X_N,$$

where $C_3, C_4, C_5 > 0$ are absolute constants.

Entropy numbers

- Assume that $X_N \subset L_\infty(\Omega)$. Define

$$X_N^p := \{f \in X_N : \ \ \|f\|_p \le 1\}, \ \ 1 \le p < \infty.$$

- The entropy numbers of X_N^p in the space $L_\infty(\Omega)$:

$$\varepsilon_n(X_N^p, L_\infty) := inf\Big\{\varepsilon > 0 : \exists f_1, \dots, f_{2^n} \in X_N^p \\ \text{s.t. } \ X_N^p \subseteq \cup_{j=1}^{2^n} B(f_j, \varepsilon)_\infty \Big\},$$

where

$$B(f, \varepsilon)_\infty := \{g \in L_\infty(\Omega) : \ \ \|f - g\|_\infty \le \varepsilon\}.$$

Discretization in L_p: A conditional result

Theorem 1. [DPSTT1] Let $1 \le p < \infty$. Assume that $X_N \subset L_\infty$ and

$$\varepsilon_k(X_N^p, L_\infty) \le B(N/k)^{1/p} \quad \text{for all} \;\; 1 \le k \le N, \tag{4}$$

where $B \ge 2$ satisfies $\log(B) \le C_1(p)N$. Then $\forall \varepsilon \in (0,1)$, $\exists \xi^1, \cdots, \xi^m \in \Omega$ such that

$$N \le m \le C(p,\varepsilon) N B^p (\log N)^2 \;\; \text{and}$$

$$(1-\varepsilon)\|f\|_p^p \le \frac{1}{m}\sum_{j=1}^m |f(\xi^j)|^p \le (1+\varepsilon)\|f\|_p^p, \;\; \forall f \in X_N.$$

Estimates of $\varepsilon_k(X_N^p, L_\infty)$ for $1 \le p \le 2$

Theorem 2. [DPSTT2] Assume that $X_N \subset L_\infty$ satisfies

(i) there exists a constant $K_1 > 1$ such that

$$\|f\|_\infty \le (K_1 N)^{\frac{1}{2}} \|f\|_2, \quad \forall f \in X_N.$$

(ii) there exists a constant $K_2 > 1$ such that

$$\|f\|_\infty \le K_2 \|f\|_{\log N}, \quad \forall f \in X_N.$$

Then for each $1 \le p \le 2$ and $1 \le k \le N$,

$$\varepsilon_k(X_N^p, L_\infty) \le C_p (K_1 K_2^2 \log N)^{\frac{1}{p}} \left(\frac{N}{k}\right)^{\frac{1}{p}}.$$

Theorem 3. [DPSTT2] Assume that $X_N \subset L_\infty(\Omega)$ satisfies

$$\|f\|_\infty \le (KN)^{\frac{1}{2}} \|f\|_2, \quad \forall f \in X_N.$$

Then for any $1 \le p \le 2$ and $0 < \varepsilon < 1$, $\exists \xi^1, \cdots, \xi^m \in \Omega$ such that

$$N \le m \le C_{\varepsilon,K} N \log^3 N \quad \text{and}$$

$$(1-\varepsilon)\|f\|_p^p \le \frac{1}{m}\sum_{j=1}^m |f(\xi^j)|^p \le (1+\varepsilon)\|f\|_p^p, \quad \forall f \in X_N.$$

Comments for $p > 2$

- Denote by $\mathcal{N}(X_N; p)$ the minimum number m of required nodes ξ^j, $1 \le j \le m$ for the inequality

$$C_1 \int_\Omega |f|^p \, d\mu \le \frac{1}{m} \sum_{\nu=1}^{m} |f(\xi^\nu)|^p \le C_2 \int_\Omega |f|^p \, d\mu, \quad \forall f \in X_N$$

with $C_1, C_2 > 0$ general positive constants independent of f and N.

- Theorem 3 asserts that if $1 \le p \le 2$ and

$$\|f\|_\infty \le (KN)^{\frac{1}{2}} \|f\|_2, \quad \forall f \in X_N, \tag{5}$$

then

$$\mathcal{N}(X_N; p) \le C_K N \log^3 N. \tag{6}$$

- For $2 < p < \infty$, (5) does not imply (6).

An example

Let X_N denote the space of spherical harmonics of degree n on the unit sphere $\mathbb{S}^{d-1}$ endowed with the normalized surface Lebesgue measure. Then X_N satisfies the condition (5), but for each $2 < p < \infty$, there exists $\delta = \delta_p \in (0, 1)$ s.t.

$$\mathcal{N}(X_N; p) \geq cN^{1+\delta}.$$

Remark. Recent progress on the problem for $p > 2$ can be found in the works of Kosov, and Temlyakov.

Unconditional discretization with weights

Theorem 4. [DPSTT2] Given $1 \le p < 2$ and an arbitrary $X_N \subset L_p(\Omega)$, $\exists \xi^1, \cdots, \xi^m \in \Omega$, $\exists \lambda_1, \cdots, \lambda_m > 0$ such that

$$N \le m \le C_p N \log^3 N \ \text{ and}$$

$$\frac{1}{2}\|f\|_p \le \Big(\sum_{j=1}^m \lambda_j |f(\xi^j)|^p\Big)^{\frac{1}{p}} \le \frac{3}{2}\|f\|_p, \quad \forall f \in X_N. \tag{7}$$

Theorem 5.[DPSTT2] If $p = 2$, then the above result holds with $m \sim N$.

Part II. Discretization on C^2 domains

- $\Omega \subset \mathbb{R}^d$: a compact domain with C^2-boundary $\partial\Omega$ and the Lebesgue measure $d\mu = dx$.
- $X_N = \Pi_n^d$: the space of all algebraic polynomials in d variables of total degree at most n. Note: $N \sim n^d$.
- The results in this part are obtained in :

  [DP] F. Dai, A. Prymak, L_p-Bernstein inequalities on C^2-domains, arXiv:2010.06728 [math.NA] 12 Oct 2020.

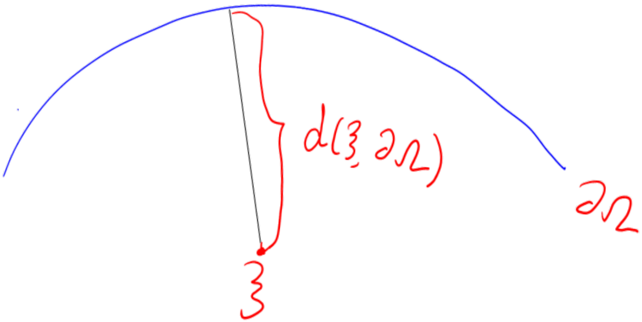
A distant function on Ω

- Let $\rho : \Omega \times \Omega \to [0, \infty)$ be the metric on Ω defined by

$$\rho(\xi, \eta) := \|\xi - \eta\| + \left| \sqrt{\mathrm{d}(\xi, \partial\Omega)} - \sqrt{\mathrm{d}(\eta, \partial\Omega)} \right|, \quad \xi, \eta \in \Omega,$$

where $\|\cdot\|$ denotes the Euclidean norm, and

$$\mathrm{d}(\xi, \partial\Omega) := \inf_{\eta \in \partial\Omega} \|\xi - \eta\|.$$



Distant functions

- In the case when $\Omega = [-1, 1]$, it can be easily seen that for $x, y \in [-1, 1]$,

$$\rho(x, y) \sim |\arccos x - \arccos y|.$$

- For $\xi \in \Omega$ and $t > 0$, define

$$B_\rho(\xi, t) := \{\eta \in \Omega : \ \ \rho(\xi, \eta) \leq t\}.$$

- It can be shown that for each $\xi \in \Omega$ and $n \in \mathbb{N}$,

$$\begin{aligned} |B_\rho(\xi, \frac{1}{n})| &\sim \inf_{\substack{P \in \Pi_n^d \\ P(\xi)=1}} \int_\Omega |P(x)|^2 \, dx \\ &\sim \left(\frac{1}{n}\right)^d \left(\sqrt{\mathrm{d}(\xi, \partial\Omega)} + \frac{1}{n}\right). \end{aligned}$$

Separated subsets of Ω

- A subset $\Lambda \subset \Omega$ is called (ρ, δ)-separated for some $\delta > 0$ if $\rho(\omega, \omega') \geq \delta$ for any two distinct pts $\omega, \omega' \in \Lambda$.

- A (ρ, δ)-separated subset $\Lambda \subset \Omega$ is called maximal if $\Omega = \bigcup_{\omega \in \Lambda} B_\rho(\omega, \delta)$.

- Given $\delta \in (0, 1)$, a maximal (ρ, δ)-separated subset $\Lambda \subset \Omega$ always exists and satisfies $\#\Lambda \sim_d \delta^{-d}$.

Discretization on Ω

Theorem.[DP] There exists a constant $\delta_0 \in (0, 1)$ depending only on Ω such that $\forall \varepsilon \in (0, \delta_0)$, $\forall 1 \leq p \leq \infty$, and every maximal $(\rho, \frac{\varepsilon}{n})$-separated subset $\{\xi^1, \cdots, \xi^m\}$ of Ω,

$$(1-\varepsilon)\|f\|_{L_p(\Omega)} \leq \Big(\sum_{j=1}^{m} |f(\xi^j)|^p \lambda_j\Big)^{\frac{1}{p}} \leq (1+\varepsilon)\|f\|_{L_p(\Omega)}, \quad \forall f \in \Pi_n^d,$$

where $\lambda_j = |B_\rho(\xi^j, \frac{\varepsilon}{n})|$, and we replace the ℓ^p-norm with $\max_{1 \leq j \leq m} |f(\xi_j)|$ if $p = \infty$.

Note here $m \sim_\varepsilon n^d \sim \dim \Pi_n^d$.

Bernstein inequality along normal and tangential directions on $\partial\Omega$

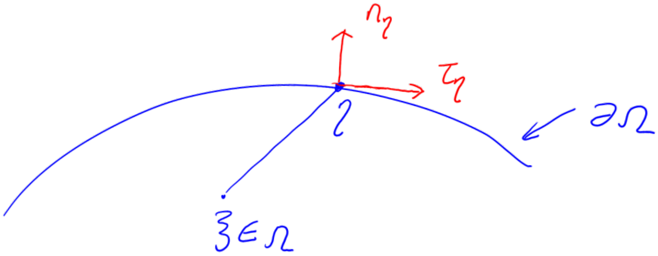
- 1D Bernstein inequality: for $f \in \Pi_n^1$ and $1 \le p \le \infty$,

$$\|\varphi^r f^{(r)}\|_{L_p[-1,1]} \le C n^r \|f\|_{L^p[-1,1]}, \quad \varphi(x) = \sqrt{1-x^2}.$$

- Higher dimensional extension: we need to replace $f^{(r)}$ with

$$\partial_{\boldsymbol{\tau}_\eta}^{\ell_1} \partial_{\mathbf{n}_\eta}^{l_2} f(\xi), \quad \ell_1, \ell_2 \in \mathbb{N}_0, \quad \xi \in \Omega,$$

where $\eta \in \partial\Omega$ lies in a "small" neighborhood of ξ, $\mathbf{n}_\eta$ ($\boldsymbol{\tau}_\eta$)is the unit normal (tangential) vector to $\partial\Omega$ at η.



Directional derivatives

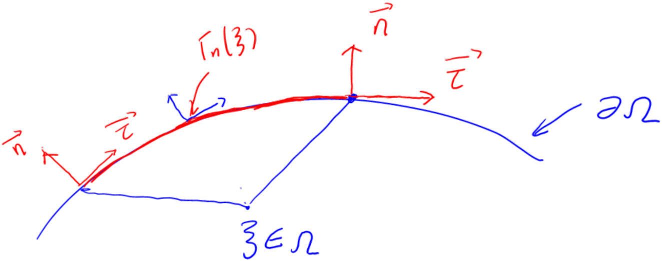
- Boundary pts associated with a pt $\xi \in \Omega$:

$$\Gamma_n(\xi) := \{\eta \in \partial\Omega : \quad \|\xi - \eta\| \le \mu\varphi_n(\xi)\},$$

where $\mu > 1$ is a parameter, and $\varphi_n(\xi) := \sqrt{\mathrm{d}(\xi, \partial\Omega)} + n^{-1}$.

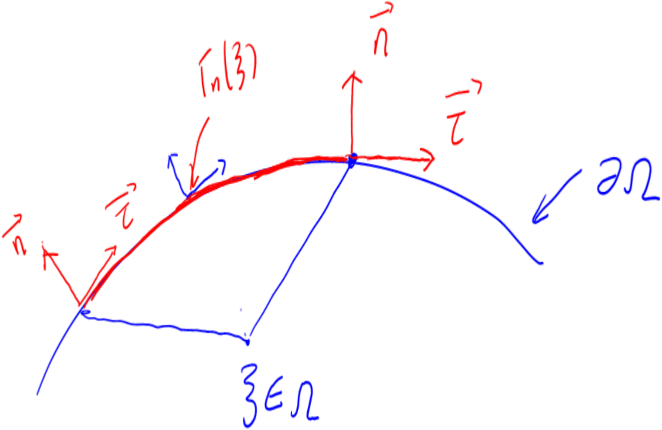
- The set of directions associated with $\xi \in \Omega$:

$$E_n(\xi) := \Big\{(\boldsymbol{\tau}_\eta, \mathbf{n}_\eta) : \quad \eta \in \Gamma_n(\xi), \ \ \boldsymbol{\tau}_\eta \cdot \mathbf{n}_\eta = 0, \ \ \|\boldsymbol{\tau}_\eta\| = 1\Big\}.$$



Directional derivatives at $\xi \in \Omega$:

$$\mathcal{D}^{l_1,l_2}_{n,\mu} f(\xi) := \max_{(\boldsymbol{\tau},\mathbf{n})\in E_n(\xi)} \left|\partial^{l_1}_{\boldsymbol{\tau}} \partial^{l_2}_{\mathbf{n}} f(\xi)\right|.$$



The L^p-Bernstein-Markov inequality

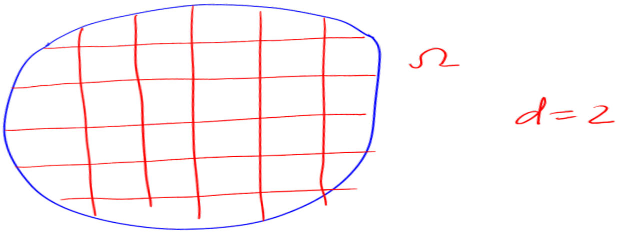
Theorem.[DP] Let $\mu \geq \sqrt{\mathrm{diam}(\Omega)}$ be a given parameter. Then for any $f \in \Pi_n^d$ and $1 \leq p \leq \infty$, we have

$$\left\| \varphi_n^j \mathcal{D}_{n,\mu}^{r,j+l} f \right\|_{L^p(\Omega)} \leq C_{\Omega,\mu} n^{r+j+2l} \|f\|_{L^p(\Omega)}, \quad r, j, l = 0, 1, \dots,$$

Moreover, the order n^{r+j+2l} here is sharp as $n \to \infty$.

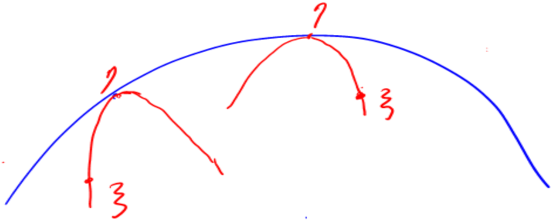
Remarks ($d = 2$)

The standard 1D Bernstein-Markov inequality applied along straight line segments from Ω would only give the order $n^{2r+j+2l}$. The improvement from $n^{2r+j+2l}$ to n^{r+j+2l} is exactly what is needed in many applications.



Key idea for $d = 2$

- Construct a family of inscribed parabolas s.t. every pt in Ω can be connected with $\partial\Omega$ via an unique parabola.



- These parabolas can be used as local coordinates to substitute the usual Cartesian coordinates s. t. each pt in Ω is covered by a local parametrization.
- A double integral over Ω can then be locally expressed as iterated integrals along these parabolas. Applying the 1D Bernstein inequality along these parabolas instead of vertical lines would yield the desired estimates.

Thank you very much !

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