Some theorems on the restriction of operator to coordinate subspace and discretization.

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I) Discretization in $C(\Omega)$

$$W \subset C(\Omega)$$

$\Omega \supset X_m = \{\xi_1, \dots, \xi_m\}$ – arbitrary set of m points. If $f \in W$

$$\|f\|_{X_m} \equiv \max\{|f(x)|,\ x \in X_m\}$$

We consider the quantity

$$D(W, m) = \inf_{X_m} \sup_{f \in W} \frac{\|f\|_{C(\Omega)}}{\|f\|_{X_m}} \tag{1}$$

For $\Lambda \subset \mathbb{Z}$

$$T(\Lambda) \equiv \operatorname{span}\{e^{ikx}, k \in \Lambda\} \subset C(-\pi, \pi) \tag{2}$$

We shall consider quantity (1) in the case when $W = T(\Lambda)$.

But first – one general result.

Th. 1 (Kashin, Temlyakov) Let L_N be an N-dimensional subspace of $C(\Omega)$
There exists a set $X_m = \{\xi_1, \ldots \xi_m\} \subset \Omega$ of $m \le 9^N$ points such that for any $f \in L_N$

$$\|f\|_{C(\Omega)} \le 2 \max_{\xi \in X_m} |f(x)|$$

Fix $b > 1$ and for $n \ge n(b)$ consider the set

$$\mathcal{K} = \{k_j\}_{j=n}^{2n-1} : \frac{k_{j+1}}{k_j} \ge b, \ j = n, \ldots, 2n-2,$$

$$\frac{k_j}{k_n} \in \mathbb{N}, \ j = n, \ldots, 2n-1$$

Fix also $\nu \le k_n/n$ and define the subspace in $C(-\pi, \pi)$:

$$T(\mathcal{K}, \nu) = \{ f : f = \sum_{j=n}^{2n-1} p_j(x) e^{ik_j x} \},$$

where $\deg p_j \le \nu$

Th 2 (K, T) 2019 For $m \ge n(2\nu+1)$

$$D(T(\mathcal{K}, \nu), m) \ge c_1 n^{1/2} \left(\ln \frac{em}{(2\nu+1)n} \right)^{-1/2}.$$

Corollary Let R_1, R_2 - absolute constant

a) $$D(T(\mathcal{K},0), m) \le R_1 \Longrightarrow$$
$$m \ge c\, e^{\delta n},$$
where $c = c(R_1) > 0$, $\delta = \delta(R_1) > 0$.

b) If $m \le R_2 n \Longrightarrow$
$$D(T(\mathcal{K},0), m) \ge c(R_2) \cdot n^{1/2},$$
$c(R_2) > 0$.

The subspace of trigonometric polynomials with fixed random spectrum.

N, n, $n \le N$, $n, N \in \mathbb{N}$

$\{\eta_k(\omega)\}_{k=-N}^{N}$ - selectors on (Ω, μ)

$(\eta_k = 0 \text{ or } 1)$ $\quad \mathbb{E}(\eta_k) = \frac{n}{2N+1}$

$$\Lambda_\omega = \{k : \eta_k(\omega) = 1\}$$

$$D(\Lambda_\omega, m) \equiv D(T(\Lambda_\omega), m)$$

Th 3 (K, T) There exists absolute constant $c > 0$ such that for $m, n, N \in \mathbb{N}$, $1 \le n \le N$ $n \le m \le N$ the expectation

$$\mathbb{E}\left(D(\Lambda_\omega, m)\right) \ge c \cdot \min\left[\frac{N}{m}, \left(\frac{n}{\log N}\right)^{1/2}\right].$$

Corollary If we have the sequence of triples $\{N_\nu, n_\nu, m_\nu\}_{\nu=1}^{\infty}$ with

$$\lim_{\nu \to \infty} n_\nu (\log N_\nu)^{-1} = \infty$$

Then if

$$\mathbb{E} D(\Lambda_\omega, m_\nu) \le K, \quad \nu = 1, 2, \dots$$

$$\Longrightarrow \quad m_\nu \ge c' N_\nu \quad \nu = 1, 2, \dots$$

$c' > 0$ - absolute constant

Discretization in the space of homogeneous polynomials in $\mathbb{C}^d$.

$P(d, n)$ - the space of homogeneous polynomials of d complex variables of degree $\le n$.

$$\dim P(d,n) \asymp n^{d-1}$$

$B^d = \{z \in \mathbb{C}^d : \|z\| \le 1\}$, $S^d = \{z \in \mathbb{C}^d : \|z\| = 1\}$,

where $\|z\| = (z,z)^{1/2}$, $(z,w) = \sum_{j=1}^{d} z_j \bar{w}_j$

We consider $P(d,n)$ as a subspace of $C(S^d)$

Th (K, 1985) For $d = 2, 3, \ldots$ there exists a constant C_d such that for $m \ge C_d \cdot \dim P(d,n)$

$$D(P(d,n), m) \le 2 \quad (3)$$

Th (Hörmander, unpublished) $p(z) \in P(d,n)$

$\|z\| = \|w\| = 1$, $(z,w) = 0$

Then

$$|(p'(z), w)| \le \left\{ n \left(1 - \frac{1}{n}\right)^{1-n} \right\}^{1/2} \|p\|_{C(S^d)} \le$$

$$\le (en)^{1/2} \|p\|_{C(S^d)}$$

Hörmander gives also a sufficient condition on Γ (instead of S^d) which guarantee the estimate analoguous to (3)

Problem Characterize the class of surfaces in $\mathbb{C}^d$ with a property similar to (3).

II) Restriction of operator to coordinate subspace.

$$T: X \to Y \quad \text{operator}$$

X or Y has a basis. Main case:

X or Y – finite dimensional normed space:

$$X = (\mathbb{R}^n, \| \ \|_X) \text{ or } Y = (\mathbb{R}^N, \| \ \|_Y)$$

The coordinate subspace – the subspace generated by some elements of a basis

The goal: to guarantee some extra properties of T by restrict it to some coordinate subspace
(Of the domain or of the range of T)

The discretization problems is a specific subdomain of this area!

Notations:

Consider $N \times n$ matrix A as the operator from $\mathbb{R}^n$ to $\mathbb{R}^N$,

$$\|A\|_{(p,q)} = \sup_{\|x\|_{l_p^n} \leqslant 1} \|Ax\|_{l_q^N}, \ 1 \leqslant p, q \leqslant \infty$$

$$\langle N \rangle = \{1, 2, \ldots, N\}$$

$v_i, \quad i \in \langle N \rangle$ – rows of A

$w_j, \quad j \in \langle n \rangle$ – columns of A

If $\Omega \subset \langle N \rangle$, then $A(\Omega)$ – submatrix generated by v_i, $i \in \Omega$

$(\cdot\,, \cdot)$ – scalar product in $\mathbb{R}^n$

$$\|x\|_p \equiv \|x\|_{l_p^n} \quad \text{if} \quad x \in \mathbb{R}^n$$

Examples

1) $\Phi = \{\varphi_j\}_{j=1}^n \subset L^2(\Omega)$ – O.N.S.

$$\|\varphi_j\|_{L^\infty} \leq K \quad j=1,\dots,n \quad (*)$$

$$T : \ell_2^n \longrightarrow L^2(\Omega)$$

$$\{a_j\}_{j=1}^n \xrightarrow{T} \sum_{j=1}^n a_j \varphi_j \qquad \|T\| = 1$$

Th. (Bourgain, 1989) For $p>2$ there exists $\Lambda \subset \langle n \rangle$, $|\Lambda| \geq n^{2/p}$ such that

$$\|T : \ell_2^\Lambda \to L^p(\Omega)\| \leq C(K,p)$$

Th. (K., I. Limonova, 2020) Let L_{ψ_α} – Orlicz space generated by

$$\psi_\alpha(t) = t^2 \frac{\ln^\alpha(e+|t|)}{\ln^\alpha(e+1/|t|)}, \ \alpha>0$$

and $\Phi = \{\varphi_j\}_{j=1}^n$ with the property $(*)$. There exists $\Lambda \subset \langle n \rangle$, $|\Lambda| \geq n(\log n+2)^{-2\alpha}$ such that

$$\|T : \ell_\infty^\Lambda \to L_{\psi_\alpha}\| \leq C(\alpha,K)\,|\Lambda|^{1/2}.$$

2) $T: \ell_2^n \to \ell_2^N$, $\|T\| = 1$

Th (Lunin, 1989) There exists $\Lambda \subset \langle N \rangle$
$|\Lambda| = n$ such that

$$\|T: \ell_2^n \to \ell_2^\Lambda\| \le K\sqrt{\frac{n}{N}}$$

This theorem is a discretization result.

<u>Corollary</u>: If L - n-dimensional subspace in $L^2(\Omega)$, Ω - compact in $\mathbb{R}^d$ then there exists $X_n = \{\xi_1, \dots \xi_n\} \subset \Omega$ such that for any $f \in L$

$$\left(\frac{1}{n}\sum_1^n f^2(\xi_j)\right)^{1/2} \le K' \|f\|_{L^2(\Omega)}$$

(Schechtman; Marcus, Spielman, Srivastava ...)

For two sides estimates we need some extra conditions on L.

Let A $N \times n$ matrix such that (11)

$\forall x \in \mathbb{R}^n, \ \forall i_0 \in \langle N \rangle$

$$|(V_{i_0}, x)| \le \varepsilon \left(\sum_{i=1}^{N} |(V_i, x)|^q \right)^{1/q}$$

where $1 \le q < \infty$

Th. (I. Limonova, 2020) For $\varepsilon \le (rk(A))^{-1/q}$! There exists a decomposition

$$\langle N \rangle = \Omega_1 \cup \Omega_2, \quad \Omega_1 \cap \Omega_2 = \emptyset$$

such that for any $x \in \mathbb{R}^n$ and $k = 1, 2$

$$\| A(\Omega_k) x \|_q \le \gamma \| A x \|_q$$

$$\gamma = \frac{1}{2^{1/q}} + \frac{2 + 3 \cdot 2^{-1/q}}{q} \left(rk(A) \varepsilon^q \ln \frac{6q}{(rk(A) \varepsilon^q)^{1/3}} \right)^{1/3}$$

(12

If $\Phi = \{\varphi_i(x)\}_{i=1}^n$ – system of functions on X, then

$$S_\Phi^*(\{a_i\}_{i=1}^n) = f(x) = \sup_{1 \leqslant s = s(x) \leqslant n} \left| \sum_{i=1}^{s} a_i \varphi_i(x) \right|$$

Kolmogorov's rearrangement problem (1925 ? – 1930 ?):

$$\Phi = \{\varphi_i(x)\}_{i=1}^{\infty} \text{ – O.N.S.}$$

Does it exist $\sigma \in S(\infty)$ such that

$$\sum a_i \varphi_{\sigma(i)}(x)$$

converges almost everywhere if

$$\sum a_i^2 < \infty \quad ?$$

Equivalent finite-dimensional version of this problem:

$$\Phi = \{\varphi_i(x)\}_{i=1}^{n} \text{ – O.N.S.}$$

Does it exist $\sigma \in S(n)$ such that for $\Phi_\sigma = \{\varphi_{\sigma(i)}\}_{i=1}^{n}$ the operator $S^*_{\Phi_\sigma}$ has bounded weak $2 \to 2$ norm?

THEOREM 2. (J. Bourgain, 1989)

For any O.N.S. $\Phi = \{\varphi_i(x)\}_{i=1}^n$ *with*

$$\|\varphi_i\|_{L^\infty} \leqslant M, \quad i = 1, 2, \ldots, n, \qquad (*)$$

there exists $\sigma \in S(n)$ *such that*

$$\|S^*_{\Phi_\sigma} : l_2^n \to L^2\| \leqslant C_M \log\log n.$$

Almost everywhere convergence of orthogonal series and in particular Kolmogorov problem is connected with the estimates of norm of submatrices!

Classical facts:

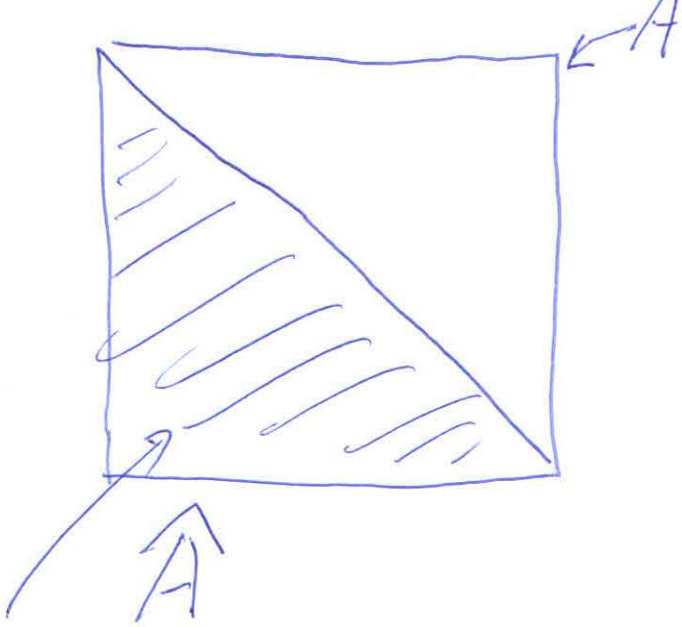
For any O.N.S. $\Phi = \{\varphi_i(x)\}_{i=1}^n$

$$\| S_\Phi^* : \ell_2^n \to L^2 \| \leq 4 \log n ;$$

and $\exists \ \Phi_0 = \{\varphi_i^0(x)\}_{i=1}^n$ such that

$$\| S_{\Phi_0}^* : \ell_2^n \to L^2 \| \geq \frac{1}{4} \log n$$

Discret version of this results:

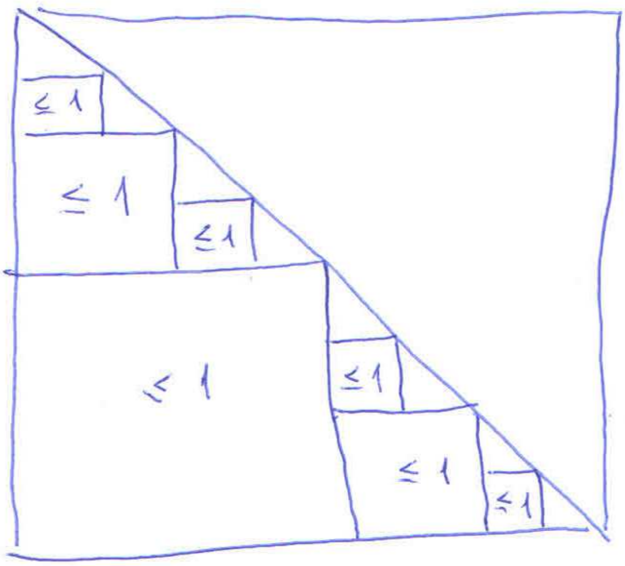


$$\| \hat{A} \| \leq 4 \log n \, \| A \|$$

and $\exists \ A_0$

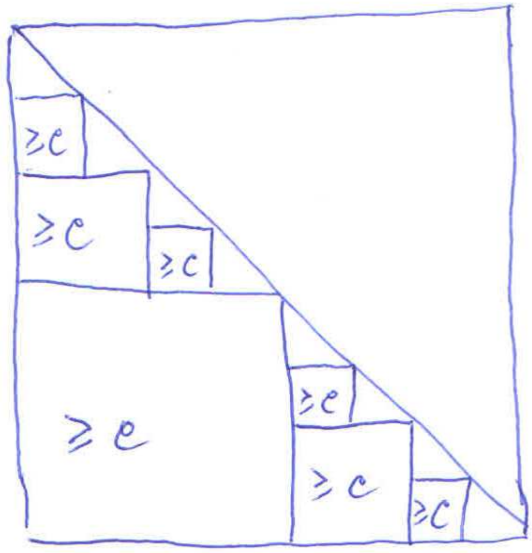
$$\| \hat{A}_0 \| \geq \frac{1}{4} \log n \, \| A \|$$

The proof of upper estimate: $\|A\|=1$



If $A = H_n$ $\quad H_n$ – Hilbert matrix

$$H_n = \{h_{ij}\},\ h_{ij} = \begin{cases} 0 & \text{if } i=j \\ \frac{1}{i-j} & \text{if } i \neq j \end{cases}$$



H_n

$c>0$ – absolute constant

This example looks very exeptional.
Important problem: give general condition on $\mathcal{F}$ to guarantee the improvement of upper $\log n$ estimate for $\|S_{\mathcal{F}}^{*} : \ell_2^n \to L_2\|$.

In 1981 I formulated a problem:

Problem: Does it exists γ_n, $n=1,2\ldots$ with $\gamma_n \to 0$ if $n \to \infty$ such that for any O.N.S of the type

$$\Psi = \{\varphi_i(x)\varphi_i(y)\}_{i=1}^{n},$$

where $\{\varphi_i\}_{i=1}^{n}$ – O.N.S.

$$\| S_{\Psi}^{*}(a) \|_{L^2} \le \gamma_n \sqrt{n} (\log n)$$

$$a = (1, 1, \ldots, 1) \quad ?$$

The problem is still open but:

Th. (G. Karagulyan, 2020, Sbornik Math, 2020 №12)

For $n=1,2,\ldots$ There exists $\sigma \in S(n)$ such that for $T_\sigma = \{e^{i\sigma(k)x}\}_{k=1}^{n}$

$$\| S_{T_\sigma}^{*} : \ell_2^n \to L^2(-\pi,\pi) \| \ge c \log n,$$

where $c > 0$ absolute constant

Corollary $\exists\ \Psi_0 = \{\varphi_i(x)\varphi_i(y)\}_{i=1}^{n}$, $n=1,2,3\ldots$ and $b = \{b_i\}_{i=1}^{n}$ $\|b\|_{\ell_2^n} = \sqrt{n}$

$$\| S_{\Psi_0}^{*}(b) \|_{L^2} \ge c' \sqrt{n} (\log n)$$

$c' > 0$ – absolute constant.