Bernstein-Markov type inequalities and discretization of norms

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1. Introduction

In the past 15-20 years the problem of discretization of uniform and L_q norms in various finite dimensional spaces has been widely investigated. In case of $L_q, 1 \leq q < \infty$ norms this problem is usually referred to as the *Marcinkiewicz-Zygmund type problem*, for uniform norm the terms *norming sets* or *optimal meshes* are used in the literature.

First discretization result for uniform norm was given by S.N. Brernstein in 1932:
For any trigonometric polynomial t_n of degree $\leq n$ and any $0 = x_0 < x_1 < ... < x_N < x_{N+1} = 2\pi$ with $x_{j+1} - x_j \leq \frac{2\sqrt{\tau}}{n}, \forall j,\ 0 < \tau < 2$ we have

$$\max_{x \in [0,2\pi]} |t_n(x)| \leq (1+\tau) \max_{0 \leq j \leq N} |t_n(x_j)|. \tag{1}$$

Thus the uniform norm of trigonometric polynomials of degree $\leq n$ can be discretized with accuracy τ using $N \sim \frac{n}{\sqrt{\tau}}$ nodes. A standard substitution leads to an extension of (1) for algebraic polynomials with nodes astisfying $\arccos x_{j+1} - \arccos x_j \leq \frac{2\sqrt{\tau}}{n}, \forall j$.

First result on the discretization of the $L_q, 1 \leq q < \infty$ norm is due to Marcinkiewicz and Zygmund, 1937:
For any univariate trigonometric polynomial t_n of degree at most n and every $1 < q < \infty$

$$\int |t_n|^q \sim \frac{1}{n} \sum_{s=0}^{2n} \left| t_n \left(\frac{2\pi s}{2n+1} \right) \right|^q \tag{2}$$

Important: the constants involved depend only on q! Number of nodes is $2n+1$.

Discretization of the L^q norms is widely applied in the study of the convergence of Fourier series, Lagrange and Hermite interpolation, positive quadrature formulas, scattered data interpolation, etc. Various generalizations were given for weighted L^q norms, multivariate polynomial on sphere and ball and general convex domains, exponential polynomials.
In terms of the methods used for the discretization several general approaches can be mentioned:

1) *Functional analytic methods*
2) *Probabilistic methods*
3) *Bernstein-Markov type inequalities*

The Bernstein-Markov method based on derivative estimates for discretized spaces always yields ***explicit*** discretization nodes. The main goal of the present talk is to give a survey of recent discretization results based on some new Bernstein-Markov type inequalities. First we will give an overview of corresponding Bernstein-Markov type inequalities. Then we will show how these Bernstein-Markov type inequalities yield new discretization results.

2. Bernstein and Markov type inequalities for derivatives of polynomials and exponential sums

2.1 Bernstein-Markov type inequalities for univariate polynomials. Markov inequality for univariate polynomials: For any algebraic polynomial q of degree $\leq n$

$$\|q'\|_{L^p_w[-1,1]} \leq c_p n^2 \|q\|_{L^p_w[-1,1]}, \quad p \geq 1. \tag{3}$$

Here L^p_w stands for the L^p norm with a *doubling* weight w. (Mastroianni and Totik)

For trigonometric polynomials t of degree $\leq n$ we have Bernstein inequality

$$\|t'\|_{L^p_w[-\pi,\pi]} \leq c_p n \|t\|_{L^p_w[-\pi,\pi]}, \quad p \geq 1, \tag{4}$$

The constants c_p depend only on p. Note: the order n^2 of derivatives in (3) in algebraic case reduces to n in trigonometric case (4). This fact makes Bernstein inequalities much more useful for obtaining discretization nodes of **asymptotically optimal cardinality**. (4) can be rewritten as

$$\|\sqrt{1-x^2}q'\|_{L^p_w[-1,1]} \leq c_p n \|q\|_{L^p_w[-1,1]}, \quad p > 0 \tag{5}$$

with q being an algebraic polynomials of degree n. Thus introduction of a weight $\sqrt{1-x^2}$ into the derivative norms reduces their size by a factor of n. This phenomena and its numerous extensions play a significant role in various discretization results.

We can unify above into a single Bernstein-Markov type inequality

$$\|(\frac{a}{n} + \sqrt{a^2-x^2})q'\|_{L^p_w[-a,a]} \leq c_p n \|q\|_{L^p_w[-a,a]}, \quad p \geq 1. \tag{6}$$

(It also holds for any trigonometric polynomial $q(t)$ of degree at most n and $0 < a < \frac{1}{2}$, Lubinsky.)

2.2 Bernstein-Markov type inequalities for multivariate polynomials. Consider the space P^d_n of real algebraic polynomials of d variables and degree at most n. The Bernstein-Markov type inequality (6) admits an extension to the multivariate case for **convex** or more generally **Lip1** domains $K \subset \mathbb{R}^d, d > 1$. The quantity $\sqrt{a^2-x^2}$ in (6) which measures the distance to the boundary of the interval in case of a convex body K can be replaced by the Hausdorff distance to the boundary $h_K(\mathbf{x}) := \inf_{\mathbf{y}\in BdK} |\mathbf{x}-\mathbf{y}|$, with Bd$K$ being the boundary of the set. This leads to the estimate

$$\| \left(\frac{1}{n} + \sqrt{h_K(\mathbf{x})}\right) \partial q\|_{L^p(K)} \leq c_{K,d} n \|q\|_{L^p(K)}, \;\; q \in P^d_n, \; 1 \leq p \leq \infty. \tag{7}$$

where ∂q stands for the gradient of q.

Note: for cuspidal domains, for instance Lipγ, $0 < \gamma < 1$ we must replace n by $n^{\frac{2}{\gamma}-1}$, and the distance to the boundary is measured differently!

The size of derivatives on the boundary of the domain plays a crucial role in deriving discretization meshes of asymptotically optimal cardinality. Therefore ***tangential*** Bernstein-Markov type inequalities are important.

Let $K \subset \mathbb{R}^d$ be a compact *star like set* that is $\mathbf{0} \in \text{Int}K$ and for every $\mathbf{x} \in K$ we have that $[\mathbf{0}, \mathbf{x}) \subset \text{Int}K$. Assume that its Minkowski functional $\varphi_K(\mathbf{x}) := \inf\{\alpha > 0 : \mathbf{x}/\alpha \in K\}$ is continuously differentiable on $\mathbb{R}^d \setminus \{\mathbf{0}\}$. Let $T_K(\mathbf{x})$ be the set of tangent unit vectors to K at $\mathbf{x} \in BdK$. Then given $0 < \alpha \leq 1$, we will say that the star like domain $K \subset \mathbb{R}^d$ is $C^{1+\alpha}$ if $\partial\varphi_K \in$Lipα.

For $C^{1+\alpha}, 0 < \alpha \leq 1$ star like domains $K \subset \mathbb{R}^d$ we have *tangential Bernstein type inequality (A.K., 2013)*

$$\|(1-\varphi_K(\mathbf{x}))^{\frac{1}{1+\alpha}-\frac{1}{2}} D_{\mathbf{u}} q\|_{L^\infty(K)} \leq c_K n \|q\|_{L^\infty(K)}, \quad q \in P_n^d, \quad \mathbf{u} \in T_K(\mathbf{x}). \tag{8}$$

If $\alpha = 0$, i.e. K is a C^1 then we have $\sqrt{1-\varphi_K(\mathbf{x})}$ in (8) which corresponds to $\sqrt{h_K(\mathbf{x})}$ in (7). On the other hand when $\alpha > 0$ the quantity $(1-\varphi_K(\mathbf{x}))^{\frac{1}{1+\alpha}-\frac{1}{2}}$ which measures the distance to the boundary in (8) gives a slower than "square root" order of decrease to 0 at the boundary. This has significant effect on decreasing the cardinality of discretization meshes.

Above tangential Bernstein type inequality requires certain smoothness of the domain. Now we present another important tangential Bernstein type inequality which holds for algebraic polynomials on any convex body in $K \subset \mathbb{R}^2$ *(A.K., 2019)*:

$$\|D_{\mathbf{u}} q\|_{L^1(BdK)} \leq c_K n \|q\|_{L^\infty(K)}, \quad q \in P_n^d, \quad \mathbf{u} \in T_K(\mathbf{x}) \tag{9}$$

Note: the size of the tangential derivative of $q \in P_n^d$ in (9) is measured in the L^1 norm on the boundary and consequently no additional weight is needed in the norm of the derivative.

2.3 Bernstein-Markov type inequalities for exponential sums. First a nice Bernstein type estimate by Borwein and Erdélyi: $\forall q(t) = \sum_{0 \leq j \leq n} c_j e^{\mu_j t}, c_j \in \mathbb{R}$ with arbitrary **real** $\mu_j \in \mathbb{R}$

$$\|(1-x^2)q'\|_{L^\infty[-1,1]} \leq (4n-2)\|q\|_{L^\infty[-1,1]}. \tag{10}$$

Surprise: this bound is independent of the exponents $\mu_j \in \mathbb{R}$ or their *degree* $\mu_n^* := \max_{0 \leq j \leq n} |\mu_j|$.
Note: derivatives are measured with the weight $1-x^2$ and not $\sqrt{1-x^2}$ as in the classical case.

A corresponding Markov type inequality states that for any $q(t) = \sum_{0 \leq j \leq n} c_j e^{\mu_j t}, c_j \in \mathbb{R}$ with arbitrary *separated* $\mu_j \in \mathbb{R}, \mu_{j+1} - \mu_j \geq 1$ of degree $\mu_n^* := \max_{0 \leq j \leq n} |\mu_j|$ we have

$$\|p'\|_{L^\infty[0,1]} \leq c n \mu_n^* \|p\|_{L^\infty[0,1]}.$$

Markov type estimate for the derivatives of multivariate exponential sums on convex bodies *(A.K. 2020)*:
$K \subset \mathbb{R}^d, d \geq 1$ is a convex body, r_K the radius of the largest inscribed ball. Then for every exponential sum

$$g(\mathbf{w}) = \sum_{0 \leq j \leq n} c_j e^{\langle \mu_j,\, \mathbf{w} \rangle}, \quad \mathbf{w}, \mu_j \in \mathbb{R}^d, \quad |\mu_k - \mu_j| \geq \frac{\delta}{r_K}, j \neq k, \ 0 < \delta \leq 1$$

we have with $\mu_n^* := \max_{0 \leq j \leq n} |\mu_n|$ and some absolute constant $c > 0$

$$\|\partial g\|_{L^\infty(K)} \leq \frac{c d^3 n^3 \mu_n^*}{\delta} \|g\|_{L^\infty(K)}. \tag{11}$$

Here only the degree $\mu_n^* := \max_{0 \leq j \leq n} |\mu_n|$ of the exponential sums, their dimension n and the *separation parameter* δ is effecting the upper bound. This important fact leads to exponent independent discretization results.

Next we give an L_p analogue of the Borwein and Erdélyi bound. Let $1 \leq p \leq 2, 0 < \delta < 1, n \in \mathbb{N}$. Then for any distinct $\lambda_1, ..., \lambda_n \in \mathbb{R}$ and any exponential sum $q(x) = \sum_{1 \leq j \leq n} c_j e^{\lambda_j x}, \forall c_j \in \mathbb{R}$ we have

$$\|(1-x^2)q'\|_{L^p[-1+\delta, 1-\delta]} \leq c \left(\ln \frac{2}{\delta}\right)^{\frac{1}{p}} n^{\frac{1}{p}+1} \|q\|_{L^p[-1,1]}. \tag{12}$$

For exponential sums with *nonnegative* coefficients a much stronger dimension and exponent independent Bernstein type upper bound can be verified *(A.K., 2021)*:

For any distinct real numbers $\lambda_j \in \mathbb{R}, 1 \leq j \leq n$ *and arbitrary exponential sum* $q(x) = \sum_{1 \leq j \leq n} a_j e^{\lambda_j x}$ *with nonnegative coefficients* $a_j \geq 0$

$$\|(1-x^2)q'\|_{L^p[-1,1]} \leq 4\|q\|_{L^p[-1,1]}, \;\; \forall p, n \in \mathbb{N}. \tag{13}$$

3. Discretization of uniform norms of polynomials and exponential sums

Now we will illustrate how Bernstein-Markov type inequalities are used in discretization of uniform norms of polynomials and exponential sums.

3.1 Discretization of uniform norms of polynomials. We define *norming sets* $Y_N \subset K$ of cardinality $\mathrm{Card} Y_N = N$for a compact set $K \subset \mathbb{R}^d$ as sets for which $\exists c_K$ with

$$||p||_{L^\infty(K)} \leq c_K ||p||_{L^\infty(Y_N)}, \;\; \forall p \in P_n^d. \tag{14}$$

Main goal: find discrete sets of possibly best asymptotic cardinality.

Since $\dim P_n^d = \binom{n+d}{n}$ we clearly must have $N > \binom{n+d}{n} \sim n^d$ in order for (14) to hold.

Optimal meshes: discrete sets of $\mathrm{Card} Y_N \sim n^d$ satisfying (14).

Finding exact geometric properties characterizing sets with optimal meshes appears to be a rather difficult problem. Using multivariate Bernstein-Markov type inequalities it was shown that C^2 star like domains and convex polytopes in $\mathbb{R}^d$ possess optimal meshes.

It was also conjectured that *any convex body in* $\mathbb{R}^d$ *possesses an optimal mesh.*

Tangential Bernstein type inequalities are especially useful in the study of optimal meshes. In particular the tangential Bernstein inequality (8) leads to the existence of optimal meshes in $C^{1+\alpha}$ star like domains with $1 - \frac{2}{d} < \alpha < 1$, *(A.K., 2013).* This is a substantial decrease in required smoothness of the star like domain compared to the C^2 condition shown earlier.

Using the tangential Bernstein type inequality (9) **the existence of optimal meshes was verified in any convex body on the plane** $\mathbb{R}^2$, *(A.K., 2019).* Recently a different proof was given by A. Prymak using an intrinsic connection between optimal meshes and asymptotic properties of the Christoffel functions.

3.2 Discretization of uniform norms of exponential sums. Now we turn our attention to some new results on discretizing uniform norms of exponential sums

$$g(\mathbf{w}) = \sum_{0\le j\le n} c_j e^{\langle \mu_j, \mathbf{w}\rangle}, \quad \mu_j, \mathbf{w} \in \mathbb{R}^d. \tag{15}$$

In contrast with the trigonometric exponential sums when the exponents $\mu_j \in \mathbb{R}^d$ in (15) are arbitrary the basis functions $e^{\langle \mu_j, \mathbf{w}\rangle}$ are in general not orthogonal, and hence this crucial Fourier analytic tool is not available. Instead we can rely again on Bernstein-Markov type inequalities of Section 2.3. As before $\mu_n^* := \max_{0\le j\le n} |\mu_j|$ is the degree of the exponential sums (15). Then we have the next discretization result *(A.K., 2020)* when $d = 1$:

Given any $n \in \mathbb{N}, 0 < \delta, \tau \le 1, [\alpha, \beta] \subset \mathbb{R},$ and $\mu_j \in \mathbb{R}, 0 \le j \le n$ satisfying $\mu_{j+1} - \mu_j \ge \frac{\delta}{\beta-\alpha}, \ \ 0 \le j \le n-1$ we can give discrete points sets $Y_N \subset [\alpha, \beta]$ of cardinality

$$N \le \frac{cn}{\sqrt{\tau}} \ln \frac{\mu_n^*}{\delta\sqrt{\tau}} \tag{16}$$

with an absolute constant $c > 0$ so that for every exponential sum $g(x) = \sum_{0\le j\le n} c_j e^{\mu_j x}, c_j \in \mathbb{R}$ we have

$$\|g\|_{L^\infty[\alpha,\beta]} \le (1+\tau)\|g\|_{L^\infty(Y_N)}.$$

An **explicit** construction of nodes is given based on equidistribution with respect to the measure

$$\mu_1(E) := \int_E \frac{dx}{1-x^2}, \quad E \subset (-1, 1) \tag{17}$$

appearing in the Bernstein type inequality (10). In addition, the discrete set is **universal** in the sense that it depends only on dimension n, degree μ_n^* and separation parameter δ of the exponential sums.

The above upper bound for the cardinality of the discrete meshes turns out to be *near optimal* in the sense that (16) is sharp with respect to both dimension n and accuracy τ up to log term, the degree μ_n^* and separation parameter δ of the exponential sums appearing only in the log term has a limited effect on the bound. In fact sharpness of the $\sqrt{\tau}$ term in the upper bound for cardinality is a special case of the following general statement *(A.K., 2020)*:

Let $K \subset \mathbb{R}^d$ be any compact set and assume that it possesses a discrete subset $Y_N \subset K$ of cardinality N so that

$$||p||_{L^\infty(K)} \le (1+\tau)||p||_{L^\infty(Y_N)}, \quad \forall p \in P_n^d.$$

Then we have some $c_K > 0$ depending only on the domain K

$$N \ge c_K \left(\frac{n}{\sqrt{\tau}}\right)^d.$$

The above discretization result extends to convex polytopes in $\mathbb{R}^d, d \ge 2$:

For any convex polytope $K \subset \mathbb{R}^d, d \ge 2$ and any $\mu_j \in \mathbb{R}^d$ satisfying $|\mu_k - \mu_j| \ge \delta, j \ne k, 0 < \delta \le 1$ we can explicitly give discrete points sets $Y_N \subset K$ of cardinality

$$N \le c(K, d) \left(\frac{n}{\sqrt{\tau}} \ln \frac{\mu_n^*}{\delta\tau}\right)^d$$

such that for every exponential sum $g(\boldsymbol{w}) = \sum_{0\le j\le n} c_j e^{\langle \mu_j, \boldsymbol{w}\rangle}, \ \boldsymbol{w} \in \mathbb{R}^d$ we have

$$\|g\|_{L^\infty(K)} \le (1+\tau)\|g\|_{L^\infty(Y_N)}.$$

4. Discretization of integral norms of polynomials and exponential sums

The following refinement of the classical Marcinkiewicz-Zygmund result similar to Bernstein's estimate (1) was recently given *(A.K., 2020)*:

For any $0 = x_0 < x_1 < ... < x_{m+1} = 2\pi$ *with spacing*

$$\max_{0\le j\le m}(x_{j+1}-x_j) < \frac{\sqrt{\tau}}{pn},$$

and every trigonometric polynomial t_n *of degree at most* n *we have*

$$(1-\tau)\sum_{j=0}^{m}\frac{x_{j+1}-x_{j-1}}{2}|t_n(x_j)|^p \le \int_0^{2\pi}|t_n(x)|^p dx \le (1+\tau)\sum_{j=0}^{m}\frac{x_{j+1}-x_{j-1}}{2}|t_n(x_j)|^p,\ \ p\ge 1. \tag{18}$$

This is a Marcinkiewicz-Zygmund type estimate of precision τ similar to Bernstein's uniform bound (1). Note that the spacing needed above can be achieved with cardinality $m \sim \frac{n}{\sqrt{\tau}}$ where this is again sharp with respect to τ, as well. In particular, choosing equidistant nodes $x_j := \frac{2\pi j}{m+1}, 0 \le j \le m+1$ with $m = \left[\frac{2\pi pn}{\sqrt{\tau}}\right] + 1$ we obtain

$$(1-\tau)\frac{2\pi}{m+1}\sum_{j=0}^{m}|t_n(x_j)|^p \le \int_0^{2\pi}|t_n(x)|^p dx \le (1+\tau)\frac{2\pi}{m+1}\sum_{j=0}^{m}|t_n(x_j)|^p.$$

We will present now a new discretization result for the integral norms of general exponential sums.

Let $1 \le p \le 2, [a,b] \subset \mathbb{R}, 0 < \delta \le 1, n \in \mathbb{N}, \Lambda > 1$. *Then we can explicitly give discrete sets* $Y_N = \{x_j\}_{j=1}^N \subset (a,b)$ *of cardinality*

$$N \le c\left(n \ln\frac{\Lambda}{\delta}\right)^{\frac{1}{p}+1}$$

with an absolute constant $c > 0$*, so that for all exponential sum* $g(x) = \sum_{0\le j\le n} c_j e^{\lambda_j x}$ *with any* $\lambda_j \in \mathbb{R}$ *satisfying*

$$\lambda_{j+1}-\lambda_j \ge \frac{\delta}{b-a}, 1 \le j \le n-1, \quad \max_{1\le j\le n}|\lambda_j| \le \Lambda$$

we have

$$\|g\|^p_{L_p([a,b])} \sim \sum_{1\le j\le N-1}(x_{j+1}-x_j)|g(x_j)|^p. \tag{19}$$

Again the estimate of the cardinality of discrete mesh is "almost" independent of the exponents λ_j since its degree Λ and separation parameter δ effects only the logarithmic term. Moreover, the discrete nodes constructed explicitly are equidistributed with respect to the measure (17).

The above discretization result admits generalization to the unit cube $I^d := [0,1]^d$ in $\mathbb{R}^d$. This requires some work because the separation condition $|\lambda_{j+1}-\lambda_j| \ge \delta, 1 \le j \le n-1$ for the exponents $\lambda_j \in \mathbb{R}^d$ of the exponential sums $g(\mathbf{w}) = \sum_{0\le j\le n} c_j e^{\langle\lambda_j,\mathbf{w}\rangle},\ \ \lambda_j, \mathbf{w} \in \mathbb{R}^d$ does not necessarily extend to projections to coordinate axises, so dimension reduction will work only with proper choice of directions.

5. Discretization of integral norms of exponential sums with nonnegative coefficients

Degree and exponent independent Bernstein-Markov type inequality (13) for exponential sums

$$g(x) = \sum_{1\le j\le n} a_j e^{\lambda_j x},\ a_j \ge 0$$

with nonnegative coefficients leads to the next Marcinkiewicz-Zygmund type result *(A.K., 2020)*:

Let $p \in \mathbb{N}, \Lambda > 1$ and consider any distinct real numbers $\lambda_j \in \mathbb{R}, 1 \le j \le n$ with $\max_{1\le j\le n} |\lambda_j| \le \Lambda$. Then discrete points sets $Y_N = \{x_1, ..., x_N\} \subset [0,1]$ of cardinality $N \le c_p \ln \Lambda$ can be given so that for every exponential sum $f(x) = \sum_{1\le j\le n} a_j e^{\lambda_j x}, a_j \ge 0$ with nonnegative coefficients we have

$$\int_0^1 f^p(x)dx \sim \sum_{1\le j\le N} (x_{j+1} - x_j) f^p(x_j). \tag{20}$$

This provides an "almost" degree independent L_p Marcinkiewicz-Zygmund type inequality for exponential sums with nonnegative coefficients in case when $p \in \mathbb{N}$ is an integer. A slight modification leads to a similar result in case of any $p \ge 1$.

Moreover the above discretization result can be extended for convex polytopes in $\mathbb{R}^d$:

Let $d, p \in \mathbb{N}, \Lambda > 1$. Consider any convex polytope $K \subset \mathbb{R}^d$. Then discrete points sets $Y_N = \{\boldsymbol{x}_1, ..., \boldsymbol{x}_N\} \subset K$ of cardinality $N = O(\ln^d \Lambda)$ and positive weights $a_1, ..., a_N$ can be given so that for any distinct $\lambda_j \in \mathbb{R}^d, 1 \le j \le n$ with $\max_{1\le j\le n} |\lambda_j| \le \Lambda$ and for every exponential sum $g(\boldsymbol{x}) = \sum_{1\le j\le n} c_j e^{\langle \lambda_j, \boldsymbol{x}\rangle}, \boldsymbol{x} \in \mathbb{R}^d, c_j \ge 0$ with nonnegative coefficients we have

$$\|g\|^p_{L_p(K)} \sim \sum_{1\le i\le N} a_i g(\mathbf{x}_i)^p.$$

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