# *Some* Methods for Proving Marcinkiewicz-Zygmund Inequalities

D. Lubinsky, Georgia Tech

- **The Rundown**

- Introduction
- Application in Lagrange Interpolation
- Nevai's Method
- The Large Sieve Method
- Duality Method for forward inequalities
- Duality Method for converse inequalities
- König's Hilbert Transform Method
- Carleson Measures
- Scaling Limits

## The Classical M-Z Inequality

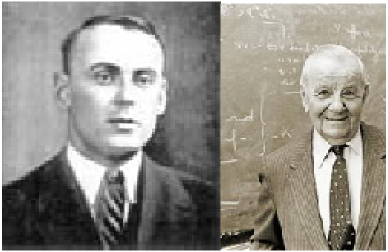
Let $P$ have degree $\leq n-1$,

*Forward Inequality: for* $p>0$,

$$\frac{1}{n}\sum_{j=1}^{n}\left|P\left(e^{2\pi ij/n}\right)\right|^{p}\leq A\int_{0}^{1}\left|P\left(e^{2\pi it}\right)\right|^{p}dt.$$

*Converse Inequality:* for $p>1$,

$$\int_{0}^{1}\left|P\left(e^{2\pi it}\right)\right|^{p}dt\leq B\left\{\frac{1}{n}\sum_{j=1}^{n}\left|P\left(e^{2\pi ij/n}\right)\right|^{p}\right\}.$$



## Application in Lagrange Interpolation

Let

$$-1 \leq x_1 < x_2 < ... < x_n \leq 1.$$

Let $L_n [f]$ interpolate $f$ at $\{x_j\}$ :

$$L_n [f] (x_j) = f (x_j) \text{ for all } j.$$

If for all polynomials $P$ of degree $\leq n - 1$, then

$$\int_{-1}^{1} |P (x)|^p w (x) dx \leq \frac{C}{n} \sum_{j=1}^{n} w (x_j) |P (x_j)|^p ,$$

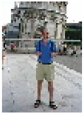
then

$$\int_{-1}^{1} |L_n [f] (x)|^p w (x) dx \leq \frac{C}{n} \sum_{j=1}^{n} w (x_j) |f (x_j)|^p$$

leading for Riemann integrable $f$, to

$$\int_{-1}^{1} |L_n [f] (x)|^p w (x) dx \leq C_1 \int_{-1}^{1} |f (x)|^p w (x) dx.$$

So Lagrange interpolation is "bounded in $L_p$".



## Nevai's Method for Forward Inequalities

Let
$$0 \leq \theta_1 < \theta_2 < \ldots < \theta_{n-1} < \theta_n < 2\pi$$
with
$$\frac{C}{n} \leq \theta_j - \theta_{j-1} \leq \frac{D}{n}.$$

Let $\deg(P) \leq An$, $p \geq 1$.

For $t \in [\theta_{j-1}, \theta_j]$, the fundamental theorem of calculus gives

$$\left|P\left(e^{i\theta_j}\right)\right|^p \leq \left|P\left(e^{it}\right)\right|^p + p\int_{\theta_{j-1}}^{\theta_j} \left|P\left(e^{is}\right)\right|^{p-1} \left|P'\left(e^{is}\right)\right| ds.$$

Then

$$\begin{aligned} & (\theta_j - \theta_{j-1}) \left|P\left(e^{i\theta_j}\right)\right|^p \\ \leq & \int_{\theta_{j-1}}^{\theta_j} \left|P\left(e^{it}\right)\right|^p dt + p\int_{\theta_{j-1}}^{\theta_j} \left|P(s)\right|^{p-1} (\theta_j - \theta_{j-1}) \left|P'(s)\right| ds. \end{aligned}$$

So

$$\begin{aligned} & \frac{C}{n} \left|P\left(e^{i\theta_j}\right)\right|^p \\ \leq & \int_{\theta_{j-1}}^{\theta_j} \left|P\left(e^{it}\right)\right|^p dt + p\frac{D}{n}\int_{\theta_{j-1}}^{\theta_j} \left|P(s)\right|^{p-1} \left|P'(s)\right| ds. \end{aligned}$$

Add:

$$\frac{C}{n}\sum_{j=1}^{n}\left|P\left(e^{i\theta_j}\right)\right|^p$$

$$\leq \int_0^{2\pi}\left|P\left(e^{it}\right)\right|^p dt+\frac{D}{n}\int_0^{2\pi}\left|P\left(e^{is}\right)\right|^{p-1}\left|P'\left(e^{is}\right)\right| ds$$

Use Hölder and then Markov-Bernstein on 2nd term:

$$\int_0^{2\pi}\left|P\left(e^{is}\right)\right|^{p-1}\left|P'\left(e^{is}\right)\right| ds$$

$$\leq \left(\int_0^{2\pi}\left|P\left(e^{is}\right)\right|^p ds\right)^{1-\frac{1}{p}}\left(\int_0^{2\pi}\left|P'\left(e^{is}\right)\right|^p ds\right)^{\frac{1}{p}}$$

$$\leq C_1 n\left(\int_0^{2\pi}\left|P\left(e^{is}\right)\right|^p ds\right)^{1-\frac{1}{p}}\left(\int_0^{2\pi}\left|P\left(e^{is}\right)\right|^p ds\right)^{\frac{1}{p}}.$$

Then

$$\frac{C}{n}\sum_{j=1}^{n}\left|P\left(e^{i\theta_j}\right)\right|^p\leq C_2\int_0^{2\pi}\left|P\left(e^{is}\right)\right|^p ds.$$

**The large sieve method: forward inequalities**

Start with a "Christoffel function" estimate: let $p > 0$. For $\theta \in [0, 2\pi]$,

$$\left| P\left(e^{i\theta}\right)\right|^p \leq Cn \int_0^{2\pi} \left| P\left(e^{it}\right)\right|^p dt. \qquad (1)$$

Let

$$K_n\left(e^{is}\right) = \sum_{j=0}^{n-1} e^{ijs}.$$

Apply (1) to $P\left(e^{it}\right) K_n\left(e^{i(t-\theta)}\right)$ :

$$\left| P\left(e^{i\theta}\right)\right|^p n^p \leq C2n \int_0^{2\pi} \left| P\left(e^{it}\right) K_n\left(e^{i(t-\theta)}\right)\right|^p dt.$$

Then if

$$0 \leq \theta_1 < \theta_2 < ... < \theta_n \leq 2\pi,$$

$$\frac{1}{n}\sum_{j=1}^{n}\left|P\left(e^{i\theta_j}\right)\right|^p \leq 2C\int_0^{2\pi}\left|P\left(e^{it}\right)\right|^p\left(\frac{1}{n^{1+p}}\sum_{j=1}^{n}\left|K_n\left(e^{i(t-\theta_j)}\right)\right|^p\right)dt.$$

If all

$$\theta_j - \theta_{j-1} \geq \frac{\varepsilon}{n}$$

then

$$\frac{1}{n}\sum_{j=1}^{n}\left|P\left(e^{i\theta_j}\right)\right|^p \leq C_1(\varepsilon)\int_0^{2\pi}\left|P\left(e^{it}\right)\right|^p dt.$$

## The Duality Method: Converse Inequalities

Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\deg(P)\le n-1$.

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|P\left(e^{it}\right)\right|^p dt\right)^{1/p}$$

$$= \sup_{\|g\|_q=1}\frac{1}{2\pi}\int_0^{2\pi}\left(\bar{g}P\right)\left(e^{it}\right)dt$$

$$= \sup_{\|g\|_q=1}\frac{1}{2\pi}\int_0^{2\pi}\left(\overline{S_n[g]}P\right)\left(e^{it}\right)dt$$

(Fourier projection)

$$= \sup_{\|g\|_q=1}\frac{1}{n}\sum_{j=1}^{n}\left(\overline{S_n[g]}P\right)\left(e^{2\pi ij/n}\right)$$

(Quadrature)

$$\le \sup_{\|g\|_q=1}\left(\frac{1}{n}\sum_{j=1}^{n}\left|S_n[g]\right|^q\left(e^{2\pi ij/n}\right)\right)^{1/q}\left(\frac{1}{n}\sum_{j=1}^{n}\left|P\left(e^{2\pi ij/n}\right)\right|^p\right)^{1/p}.$$

Now assume a forward MZ inequality, and use boundedness of the Fourier projection:

$$\begin{aligned}
& \sup_{\|g\|_q=1} \left( \frac{1}{n} \sum_{j=1}^{n} |S_n[g]|^q \left( e^{2\pi ij/n} \right) \right)^{1/q} \\
\leq \quad & C \sup_{\|g\|_q=1} \left( \int_0^{2\pi} \left| S_n[g] \left( e^{it} \right) \right|^q dt \right)^{1/q} \\
\leq \quad & C \sup_{\|g\|_q=1} \left( \int_0^{2\pi} \left| g \left( e^{it} \right) \right|^q dt \right)^{1/q} = C.
\end{aligned}$$

So

$$\left( \frac{1}{2\pi} \int_0^{2\pi} \left| P \left( e^{it} \right) \right|^p dt \right)^{1/p} \leq C \left( \frac{1}{n} \sum_{j=1}^{n} \left| P \left( e^{2\pi ij/n} \right) \right|^p \right)^{1/p}.$$

## The Duality Method: Forward Inequalities

$$\begin{aligned} & \left( \frac{1}{n} \sum_{j=1}^{n} \left| P\left( e^{2\pi i j/n} \right) \right|^{p} \right)^{1/p} \\ = & \sup_{R} \frac{1}{n} \sum_{j=1}^{n} P\left( e^{2\pi i j/n} \right) \overline{R\left( e^{2\pi i j/n} \right)} \\ = & \sup_{R} \frac{1}{2\pi} \int_{0}^{2\pi} P\left( e^{it} \right) \overline{R\left( e^{it} \right)} dt \\ \leq & \left( \frac{1}{2\pi} \int_{0}^{2\pi} \left| P\left( e^{it} \right) \right|^{p} dt \right)^{1/p} \sup_{R} \left( \frac{1}{2\pi} \int_{0}^{2\pi} \left| R\left( e^{it} \right) \right|^{q} dt \right)^{1/q}. \end{aligned}$$

Here assuming a converse inequality,

$$\begin{aligned} & \sup \left( \frac{1}{2\pi} \int_{0}^{2\pi} \left| R\left( e^{it} \right) \right|^{q} dt \right)^{1/q} \\ \leq & \; C \sup \left( \frac{1}{n} \sum_{j=1}^{n} \left| R\left( e^{2\pi i j/n} \right) \right|^{q} \right)^{1/q} = C. \end{aligned}$$

## Hermann König's method for converse inequalities (1990's)

This uses Hilbert transforms

$$H[f](x) = PV \int_{-\infty}^{\infty} \frac{f(t)}{t-x} dt.$$

Let

$$-1 \leq x_1 < x_2 < ... < x_{n-1} < x_n \leq 1$$

and

$$p_n(x) = \prod_{j=1}^{n} (x - x_j)$$

$$I_j = [x_{j-1}, x_j].$$

Approximate:

$$\frac{1}{x - x_j} \approx \frac{1}{|I_j|} H\left[\chi_{I_j}\right](x).$$

By Lagrange, for $\deg(P) < n$,

$$P(x) = p_n(x) \sum_{j=1}^{n} \frac{P(x_j)}{p_n'(x_j)(x - x_j)} = p_n(x) \sum_{j=1}^{n} \frac{y_j}{x - x_j}.$$

Then

$$\begin{aligned} P(x) &= p_n(x) \left\{ \sum_{j=1}^{n} y_j \left( \frac{1}{x - x_j} - \frac{1}{|I_j|} H\left[\chi_{I_j}\right](x) \right) \right\} \\ &\quad + p_n(x) H\left[ \sum_{j=1}^{n} y_j \frac{\chi_{I_j}}{|I_j|} \right](x) \\ &=: J_1(x) + J_2(x). \end{aligned}$$

Then

$$\|P\|_{L_p[-1,1]} \le \|J_1\|_{L_p[-1,1]} + \|J_2\|_{L_p[-1,1]}$$

Let $1 < p < 4$. Here for nice $p_n$

$$\begin{aligned} & \left\| p_n(x) H \left[ \sum_{j=1}^{n} y_j \frac{\chi_{I_j}}{|I_j|} \right] (x) \right\|_{L_p[-1,1]} \\ \leq \quad & C \left\| (1-|x|)^{-1/4} H \left[ \sum_{j=1}^{n} y_j \frac{\chi_{I_j}}{|I_j|} \right] (x) \right\|_{L_p[-1,1]} \\ \leq \quad & C \left\| (1-|x|)^{-1/4} \left[ \sum_{j=1}^{n} y_j \frac{\chi_{I_j}}{|I_j|} \right] (x) \right\|_{L_p[-1,1]} \\ \leq \quad & \ldots \leq C \left( \sum_{j=1}^{n} \frac{1}{n} \left(1-x_j^2\right)^{1/2} |P(x_j)|^p \right)^{1/p}. \end{aligned}$$

Estimation of

$$\left\| p_n(x) \left\{ \sum_{j=1}^{n} y_j \left( \frac{1}{x - x_j} - \frac{1}{|I_j|} H\left[ \chi_{I_j} \right](x) \right) \right\} \right\|_{L_p[-1,1]}$$

is difficult. It is first estimated by a quadrature sum, then split into three terms, then estimated using non-trivial matrix operator bounds. At the end we get

$$\|P\|_{L_p[-1,1]} \leq \left( \sum_{j=1}^{n} \frac{1}{n} \sqrt{1 - x_j^2} \, |P(x_j)|^p \right)^{1/p}.$$

**Theorem of König-Nielsen for Jacobi Weights** Let

$$w(x) = (1-x)^{\alpha} (1+x)^{\beta}.$$

Let

$$\begin{aligned} A &= \max\left\{1, \frac{4(\alpha+1)}{2\alpha+5}, \frac{4(\beta+1)}{2\beta+5}\right\}; \\ B &= \max\left\{1, \frac{4(\alpha+1)}{2\alpha+3}, \frac{4(\beta+1)}{2\beta+3}\right\}; \\ M &= \frac{B}{B-1}. \end{aligned}$$

The following are equivalent:
(I) For $n \geq 1$ and degree$(P) < n$,

$$\left(\int_{-1}^{1} |P|^p w\right)^{1/p} \leq C \left[\sum_{j=1}^{n} \lambda_{jn} |P(x_{jn})|^p\right]^{1/p},$$

where $\{x_{jn}\}, \{\lambda_{jn}\}$ are Gauss quadrature points and weights for $w$.
(II)

$$A < p < M.$$

## Carleson Measures for forward inequalities (Zhong, Zhu, ...)

Let $P$ be real, $p \geq 1$. From Cauchy, for real $x, \varepsilon_x > 0$,

$$|P(x)|^p \leq \frac{1}{\pi} \int_0^\pi \left| P\left(x + \varepsilon_x e^{i\theta}\right) \right|^p d\theta.$$

Then

$$\begin{aligned} & \sum_{j=1}^{n} \frac{1}{n} \sqrt{1 - x_j^2}\, |P(x_j)|^p \\ \leq \; & \sum_{j=1}^{n} \frac{1}{n} \sqrt{1 - x_j^2}\, \frac{1}{\pi} \int_0^\pi \left| P\left(x + \varepsilon_x e^{i\theta}\right) \right|^p d\theta \\ = \; & \int |P(z)|^p d\sigma_n(z). \end{aligned}$$

Show $\sigma_n$ is a *Carleson measure* in $\mathrm{Im}\, z \geq 0$, estimate its Carleson norm $N(\sigma_n)$. Replace $P$ by $G$, analytic in $\mathrm{Im}\, z > 0$, with $G_+(x) = P(x)$, $x \in (-1, 1)$:

$$\begin{aligned} & \int |P(z)|^p d\sigma_n(z) \\ \leq \quad & C_1 \int |G(z)|^p d\sigma_n(z) \\ \leq \quad & C_1 N(\sigma_n) \int_{-\infty}^{\infty} |G(x)|^p \, dx \\ \leq \quad & C_2 \int_{-1}^{1} |P(x)|^p \, dx. \end{aligned}$$

Deduce

$$\sum_{j=1}^{n} \frac{1}{n} \sqrt{1 - x_j^2} \, |P(x_j)|^p \leq C \int_{-1}^{1} |P(t)|^p \, dt.$$

## Scaling Limits: polynomials to functions of exponential type $\leq \pi$

Show the MZ inequalities

$$A \leq \int_0^1 \left| P\left(e^{2\pi it}\right)\right|^p dt / \left\{ \frac{1}{n} \sum_{j=1}^{n} \left| P\left(e^{2\pi ij/n}\right)\right|^p \right\} \leq B.$$

are equivalent to the Plancherel-Polya Inequalities 

$$A \leq \int_{-\infty}^{\infty} \left| f(t) \right|^p dt / \left\{ \sum_{j=-\infty}^{\infty} \left| f(j) \right|^p \right\} \leq B.$$

**The Idea**

Let $\{\ell_{jn}\}$ be fundamental polynomials at *n*th roots of unity:

$$\ell_{jn}(z) = \frac{1}{n} \frac{z^n - 1}{ze^{-2\pi ij/n} - 1}$$

Then

$$\lim_{n \to \infty} \ell_{jn}\left(e^{2\pi it/n}\right) = e^{i\pi t} (-1)^j \mathbb{S}(t-j)$$

where $\mathbb{S}(t) = \frac{\sin \pi t}{\pi t}$.

**Sampling at Bessel zeros**

**Theorem (2017)**

Under appropriate restrictions on $\alpha, \sigma, p$

$$A \leq \int_{-\infty}^{\infty} |f(t)|^p |t|^{2\alpha+2\sigma+1} dt / \sum_{k=1}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f(j_k)|^p \leq B$$

for all even entire functions $f$ of exponential type $\leq 1$.

## Some References


- H. König, *Vector-Valued Lagrange Interpolation and Mean Convergence of Hermite Series*, Functional analysis (Essen, 1991), 227–247, Marcel Dekker, Lecture Notes in Pure and Appl. Math., 150 (227-247).
- H. König and N. Nielsen, *Vector-Valued $L_p$ convergence of orthogonal series and Lagrange interpolation*, Forum Mathematicum, 6(1994), 183-207.
- D. Lubinsky, *On Sharp Constants in Marcinkiewicz-Zygmund and Plancherel-Polya Inequalities*, Proc.Amer. Math. Soc., 142(2014), 3575-3584.
- P. Nevai, *Geza Freud, Orthogonal Polynomials and Christoffel Functions*. A Case Study, J. Approx. Th., 48(1986), 3-167.
- J.Ortega-Cerda, J. Saludes, *Marcinkiewicz-Zygmund inequalities*, J. Approx. Th., 145(2007), 237-252.
- L. Zhong and L. Zhu, *The Marcinkiewicz-Zygmund Inequality on a smooth simple arc*, J. Approx. Theory, 83(1995), 65-83.
- A. Zygmund, *Trigonometric Series*, Cambridge University Press.


THANK YOU FOR YOUR ATTENTION
PLEASE CLAP AND DON'T ASK TOUGH QUESTIONS
...emegenerator.net