Christoffel function and discretization

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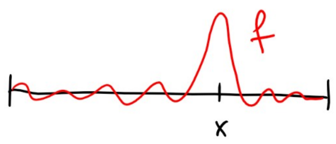
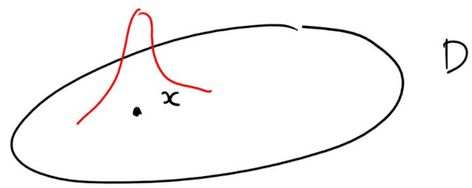
Based primarily on the pre-print "Geometric computation of Christoffel function on planar convex domains"
http://arxiv.org/abs/2003.12833

Let $D \subset \mathbb{R}^d$ be a compact set with non-empty interior, $\Pi_{n,d}$ be the space of real algebraic polynomials of total degree $\leq n$ in d variables. Equip D with Lebesgue measure and let $\{p_j\}_{j=1}^N$ be an orthonormal basis of $\Pi_{n,d}$ with respect to the inner product $\langle f, g \rangle = \int_D f g \, d\boldsymbol{x}$, $N = \dim(\Pi_{n,d}) = \binom{n+d}{d}$. Christoffel function associated with D is then

$$\lambda_n(\boldsymbol{x}, D) := \left(\sum_{j=1}^N p_j(\boldsymbol{x})^2 \right)^{-1}.$$

A useful equivalent definition is $$\lambda_n(\boldsymbol{x}, D) = \min_{f \in \Pi_{n,d},\, |f(\boldsymbol{x})|=1} \|f\|^2_{L^2(D)}, \quad \boldsymbol{x} \in D.$$





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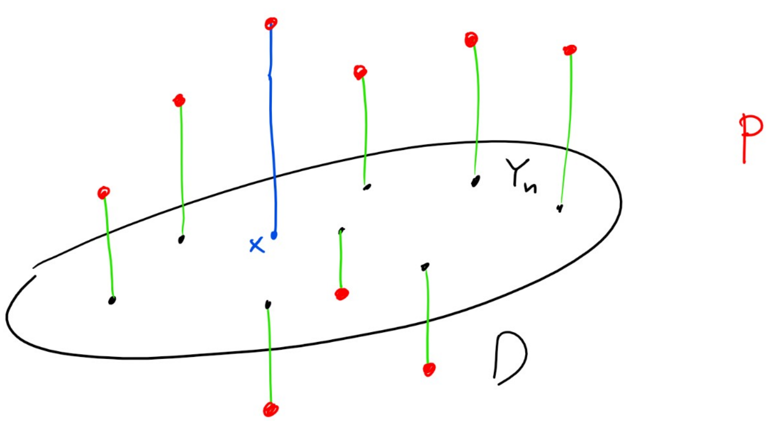
For a compact set $D \subset \mathbb{R}^d$ with non-empty interior and a continuous function f on D, we denote $\|f\|_{C(D)} = \max_{\boldsymbol{x} \in D} |f(\boldsymbol{x})|$. If there exists a sequence $\{Y_n\}_{n \geq 1}$ of finite subsets of D such that the cardinality of Y_n is at most μn^d while

$$\|p\|_{C(D)} \leq \nu \|p\|_{C(Y_n)} \quad \text{for any} \quad p \in \Pi_{n,d},$$

where $\mu, \nu > 0$ are constants depending only on D, then D *possesses optimal polynomial meshes.*



2

It was conjectured by Kroo [K1] that any convex compact set possesses optimal polynomial meshes. Until recently, this was established only for various classes of domains, namely, for convex polytopes in [K1], for C^α star-like domains with $\alpha > 2 - \frac{2}{d}$ in [K2], for certain extension of C^2 domains in [P1]. Finally, in [K4] Kroo settled the conjecture in affirmative for $d = 2$ proving existence of optimal polynomial meshes for arbitrary planar convex domains using certain tangential Bernstein inequality. For $d \geq 3$ the question is still open. Here we show another proof of the conjecture for $d = 2$ using a different technique based on Christoffel functions and an application of Tchakaloff's theorem.

[K1] András Kroó, *On optimal polynomial meshes*, J. Approx. Theory **163** (2011), no. 9, 1107–1124.

[K2] ———, *Bernstein type inequalities on star-like domains in $\mathbb{R}^d$ with application to norming sets*, Bull. Math. Sci. **3** (2013), no. 3, 349–361.

[P1] Federico Piazzon, *Optimal polynomial admissible meshes on some classes of compact subsets of $\mathbb{R}^d$*, J. Approx. Theory **207** (2016), 241–264.

[K4] András Kroó, *On the existence of optimal meshes in every convex domain on the plane*, J. Approx. Theory **238** (2019), 26–37.

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[BV] Len Bos and Marco Vianello, *Tchakaloff polynomial meshes*, Ann. Polon. Math. **122** (2019), no. 3, 221–231.

Lemma 4.1 ([BV, Lemma 2.2]). *Suppose $X_{2n} = \{\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(s)}\} \subset D$ are the nodes of a positive quadrature formula precise for $\Pi_{2n,d}$, i.e. there exist weights $w_i > 0$, $i = 1, \dots, s$, such that*

$$\int_D p(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{i=1}^{s} w_i p(\boldsymbol{x}^{(i)}) \quad \forall p \in \Pi_{2n,d}. \tag{4.1}$$

Then for any $\boldsymbol{\xi} \in D$

$$|p(\boldsymbol{\xi})| \leq \sqrt{\frac{\lambda_n(\boldsymbol{\xi}, D)}{\lambda_{2n}(\boldsymbol{\xi}, D)}}\, \|p\|_{C(X_{2n})} \quad \forall p \in \Pi_{n,d}.$$

$$\lambda_n(\boldsymbol{x}, D) = \min_{f \in \Pi_{n,d},\, |f(\boldsymbol{x})|=1} \|f\|^2_{L^2(D)}, \quad \boldsymbol{x} \in D. \tag{1.2}$$

Proof. Fix $\boldsymbol{\xi} \in D$. Let $q \in \Pi_{n,d}$ be a polynomial attaining the minimum in (1.2), i.e.,

$$q(\boldsymbol{\xi}) = 1 \quad \text{and} \quad \int_D q^2(\boldsymbol{x})\, d\boldsymbol{x} = \lambda_n(\boldsymbol{\xi}, D). \tag{4.2}$$

For any $p \in \Pi_{n,d}$, define $r(\boldsymbol{x}) := p(\boldsymbol{x})q(\boldsymbol{x})$, $\boldsymbol{x} \in D$, then $r \in \Pi_{2n,d}$. Further, by (1.2)

$$p^2(\boldsymbol{\xi}) = r^2(\boldsymbol{\xi}) \leq \lambda_{2n}^{-1}(\boldsymbol{\xi}, D) \int_D r^2(\boldsymbol{x})\, d\boldsymbol{x}, \tag{4.3}$$

while by (4.1) and (4.2)

$$\int_D r^2(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{i=1}^{s} w_i p^2(\boldsymbol{x}^{(i)}) q^2(\boldsymbol{x}^{(i)}) \leq \|p\|^2_{C(X_{2n})} \sum_{i=1}^{s} w_i q^2(\boldsymbol{x}^{(i)}) = \|p\|^2_{C(X_{2n})} \lambda_n(\boldsymbol{\xi}, D),$$

which, in combination with (4.3), is the required inequality. $\square$

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Existence of the required positive quadrature formula (4.1) with $s \le \dim(\Pi_{2n,d})$ is well-known. For the Lebesgue measure, which is our settings, this was originally proved by Tchakaloff [T]. The result has been generalized in various directions, see, for example [P] and [DPTT, Theorem 4.1].

[T] Vladimir Tchakaloff, *Formules de cubatures mécaniques à coefficients non négatifs*, Bull. Sci. Math. (2) **81** (1957), 123–134 (French).

[P] Mihai Putinar, *A note on Tchakaloff's theorem*, Proc. Amer. Math. Soc. **125** (1997), no. 8, 2409–2414.

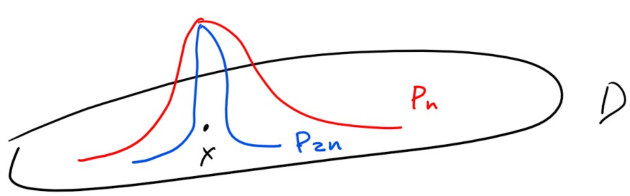
[DPTT] F. Dai, A. Prymak, V. N. Temlyakov, and S. Yu. Tikhonov, *Integral norm discretization and related problems*, Uspekhi Mat. Nauk **74** (2019), no. 4(448), 3–58 (Russian, with Russian summary); English transl., Russian Math. Surveys **74** (2019), no. 4, 579–630.

By Tchakaloff's theorem and Lemma 4.1, we obtain the following.

Proposition 4.2. *Suppose $D \subset \mathbb{R}^d$ is a compact set with non-empty interior satisfying*

$$\lambda_n(\boldsymbol{x}, D) \le c(D)\lambda_{2n}(\boldsymbol{x}, D) \quad \textit{for any} \quad \boldsymbol{x} \in D$$

with $c(D) > 0$ independent of n and $\boldsymbol{x}$. Then D possesses optimal polynomial meshes.



This proposition in combination with Corollary 3.5 immediately implies existence of optimal polynomial meshes for arbitrary planar convex domains.

Corollary 3.5. *For any planar convex domain D, $\boldsymbol{x} \in D$ and $n \ge 1$*

$$\lambda_{2n}(\boldsymbol{x}, D) \approx \lambda_n(\boldsymbol{x}, D). \tag{3.2}$$

Remark 4.5. Slight changes of arguments presented here allow to obtain an ε-version of the existence of optimal meshes. Namely, for every planar convex body D and every $\varepsilon > 0$ there exists a sequence $\{Y_n\}_{n\geq 1}$ of finite subsets of D such that the cardinality of Y_n is at most $\mu(\varepsilon)n^2$ while

$$\|p\|_{C(D)} \leq (1+\varepsilon)\|p\|_{C(Y_n)} \quad \text{for any} \quad p \in \Pi_{n,2}.$$

To achieve this, one simply needs to consider $\lambda_{mn}(\boldsymbol{\xi}, D)$ in place of $\lambda_{2n}(\boldsymbol{\xi}, D)$ for a sufficiently large $m = m(\varepsilon)$.

Remark 4.6. Note that Tchalakoff's points can be found numerically, see e.g. [D].

[D] Philip J. Davis, *A construction of nonnegative approximate quadratures*, Math. Comp. **21** (1967), 578–582.

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Typically, *asymptotics* of Christoffel function is established showing that for any fixed point $\boldsymbol{x}$ in the interior of D one has $\lim_{n\to\infty} n^d\lambda_n(\boldsymbol{x}, D) = \Psi(\boldsymbol{x})$ for an explicit or estimated function $\Psi(\boldsymbol{x})$, in which case the decay of $\Psi(\boldsymbol{x})$ when $\boldsymbol{x}$ is close to the boundary of D is of particular interest. We establish *behavior* of Christoffel function, i.e., for any n and any $\boldsymbol{x} \in D$ we calculate $\lambda_n(D, \boldsymbol{x})$ up to a constant factor independent of n and $\boldsymbol{x}$. This implies estimates of $\Psi(\boldsymbol{x})$ (provided it exists) and is useful in applications where n is fixed while $\boldsymbol{x}$ varies.

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$$\lambda_n(\boldsymbol{x}, D) := \left(\sum_{j=1}^{N} p_j(\boldsymbol{x})^2 \right)^{-1}. \tag{1.1}$$

For specific domains, such as simplex, cube or ball, an orthonormal basis can be computed and (1.1) can be used to find Christoffel function, see, e.g. [X].

[X] Yuan Xu, *Asymptotics for orthogonal polynomials and Christoffel functions on a ball*, Methods Appl. Anal. **3** (1996), no. 2, 257–272.

$$\lambda_n(\boldsymbol{x}, D) = \min_{f \in \Pi_{n,d},\, |f(\boldsymbol{x})|=1} \|f\|^2_{L^2(D)}, \quad \boldsymbol{x} \in D. \tag{1.2}$$

By (1.2), for two domains satisfying $D_1 \subset D_2 \subset \mathbb{R}^2$ $\qquad \lambda_n(\boldsymbol{x}, D_1) \le \lambda_n(\boldsymbol{x}, D_2), \quad \boldsymbol{x} \in D_2,$

and for any $\mathcal{T} \in \mathbb{A}$ $\qquad \lambda_n(\mathcal{T}\boldsymbol{x}, \mathcal{T}D) = \lambda_n(\boldsymbol{x}, D)|\det \mathcal{T}|, \quad \boldsymbol{x} \in D.$

In [K3] lower and upper estimates on Christoffel function on convex and starlike domains were obtained in terms of Minkowski functional of the body.

[K3] András Kroó, *Christoffel functions on convex and starlike domains in $\mathbb{R}^d$*, J. Math. Anal. Appl. **421** (2015), no. 1, 718–729.

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In [P2] we obtained upper estimates on Christoffel function for convex domains in terms of few easy-to-measure geometric characteristics of the location of $\boldsymbol{x}$ inside D. The estimates were obtained comparing D with a parallelotop containing D. This was followed by the lower estimates in [PU1] obtained by comparison with ellipsoids contained in D. In particular, in [PU1] the behavior of Christoffel function was computed for $\{(x_1, x_2) : |x_1|^\alpha + |x_2|^\alpha \le 1\}$ if $1 < \alpha < 2$ and it was suggested that the class of convex bodies for which the lower bounds of [PU1] and upper bounds of [P2] match (up to a constant factor) is rather large.

[P2] A. Prymak, *Upper estimates of Christoffel function on convex domains*, J. Math. Anal. Appl. **455** (2017), no. 2, 1984–2000.

[PU1] A. Prymak and O. Usoltseva, *Pointwise behavior of Christoffel function on planar convex domains*, in: Topics in classical and modern analysis. In memory of Yingkang Hu, Birkhäuser, 2019, pp. 293–302, available at `arXiv:math.CA/1709.10509`.

In this work, we establish characterization of the behavior of Christoffel function on *arbitrary planar convex domains* using comparison with ellipses contained in the domain for the lower bound and comparison with parallelepipeds containing the domain for the upper bound. This is achieved by an appropriate refinement of the ideas of [P2] and [PU1].

The proofs are constructive and explicitly describe required ellipse and parallelepiped.

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Let $D \subset \mathbb{R}^2$ be a convex body, i.e. convex compact set with $\operatorname{int}(D) \neq \emptyset$. For each $\boldsymbol{x} \in \operatorname{int}(D)$, define

$B = \{x : \|x\| \leq 1\}$
$S = [0,1]^2$
$y = ((y)_1, (y)_2)$
$\mathbb{A}$ - affine transforms

$$L(\boldsymbol{x}, D) := \sup\{(1 - \|\mathcal{L}^{-1}\boldsymbol{x}\|)^{1/2} |\det \mathcal{L}| : \mathcal{L} \in \mathbb{A},\ \boldsymbol{x} \in \mathcal{L}B \subset D\} \tag{2.1}$$

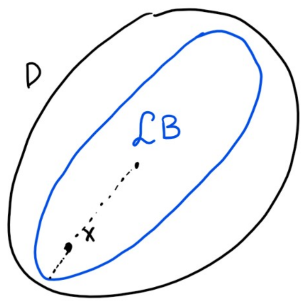
and

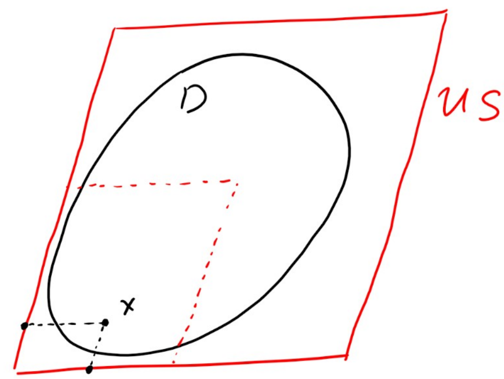
$$U(\boldsymbol{x}, D) := \inf\{((\mathcal{U}^{-1}\boldsymbol{x})_1 (\mathcal{U}^{-1}\boldsymbol{x})_2)^{1/2} |\det \mathcal{U}| : \mathcal{U} \in \mathbb{A},\ \boldsymbol{x} \in \mathcal{U}(\tfrac{1}{2}S),\ D \subset \mathcal{U}S\}. \tag{2.2}$$

Theorem 2.1. *For any planar convex body D and any interior point $\boldsymbol{x} \in D$*

$$U(\boldsymbol{x}, D) \leq cL(\boldsymbol{x}, D), \tag{2.3}$$

where c is an absolute constant.





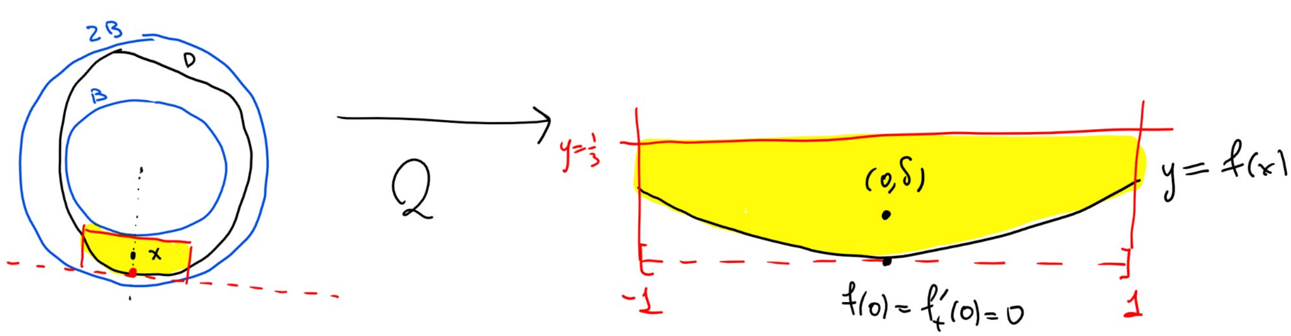
John's theorem on inscribed ellipsoid of largest volume $\quad B \subset D \subset 2B$.

Lemma 2.2. *Suppose $\boldsymbol{x} \in \operatorname{int}(D)$, $\boldsymbol{x} \neq \boldsymbol{0}$, and $\delta > 0$ are such that $(1 + \delta/\|\boldsymbol{x}\|)\boldsymbol{x} \in \partial D$. Then there exist a convex function $f : [-1, 1] \to [0, \frac{1}{3}]$ such that $f(0) = f'_+(0) = 0$ and $|f'_\pm(x)| \leq 2$ for $x \in [-1, 1]$, and an affine transform $\mathcal{Q} \in \mathbb{A}$ with $\det \mathcal{Q} = 3$ such that $\mathcal{Q}\boldsymbol{x} = (0, \delta)$,*

$$(\mathcal{Q}D) \cap ([-1, 1] \times [0, \tfrac{1}{3}]) = \{(x, y) : -1 \leq x \leq 1,\ f(x) \leq y \leq \tfrac{1}{3}\}, \tag{2.5}$$

and

$$(\mathcal{Q}\partial D) \cap ([-1, 1] \times [0, \tfrac{1}{3}]) = \{(x, y) : -1 \leq x \leq 1,\ y = f(x)\}. \tag{2.6}$$

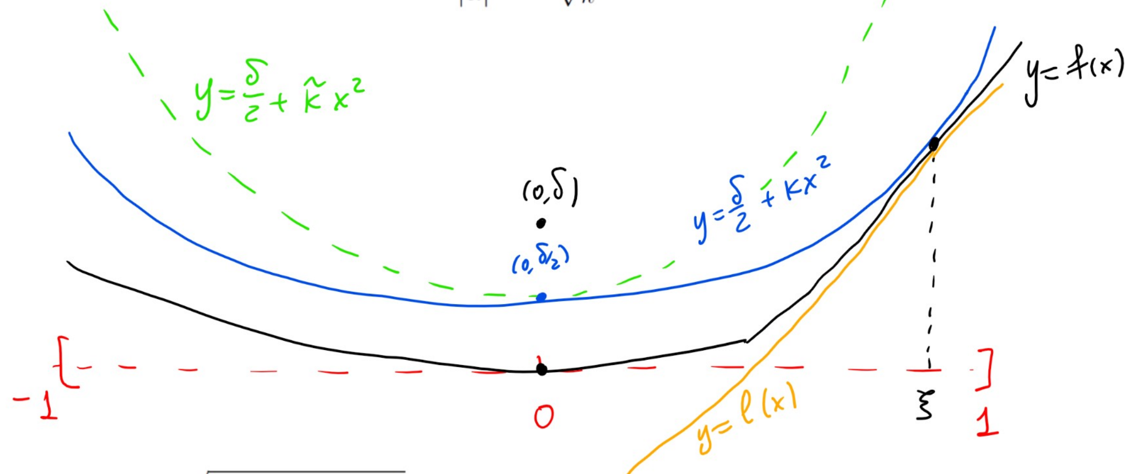


Lemma 2.4. *Suppose $f : [-1,1] \to [0, \frac{1}{3}]$ is a convex function such that $f(0) = f'_+(0) = 0$ and $|f'_\pm(x)| \le 2$ for $x \in [-1,1]$. Assume, in addition, that $0 < \frac{\delta}{2} < f(-1) + f(1)$. Then there exist $k > 0$, $\xi \in [-1,1] \setminus \{0\}$, and a linear function $\ell(x) = \alpha x - \beta$ with $|\alpha|, \beta \in (0,2]$, such that*

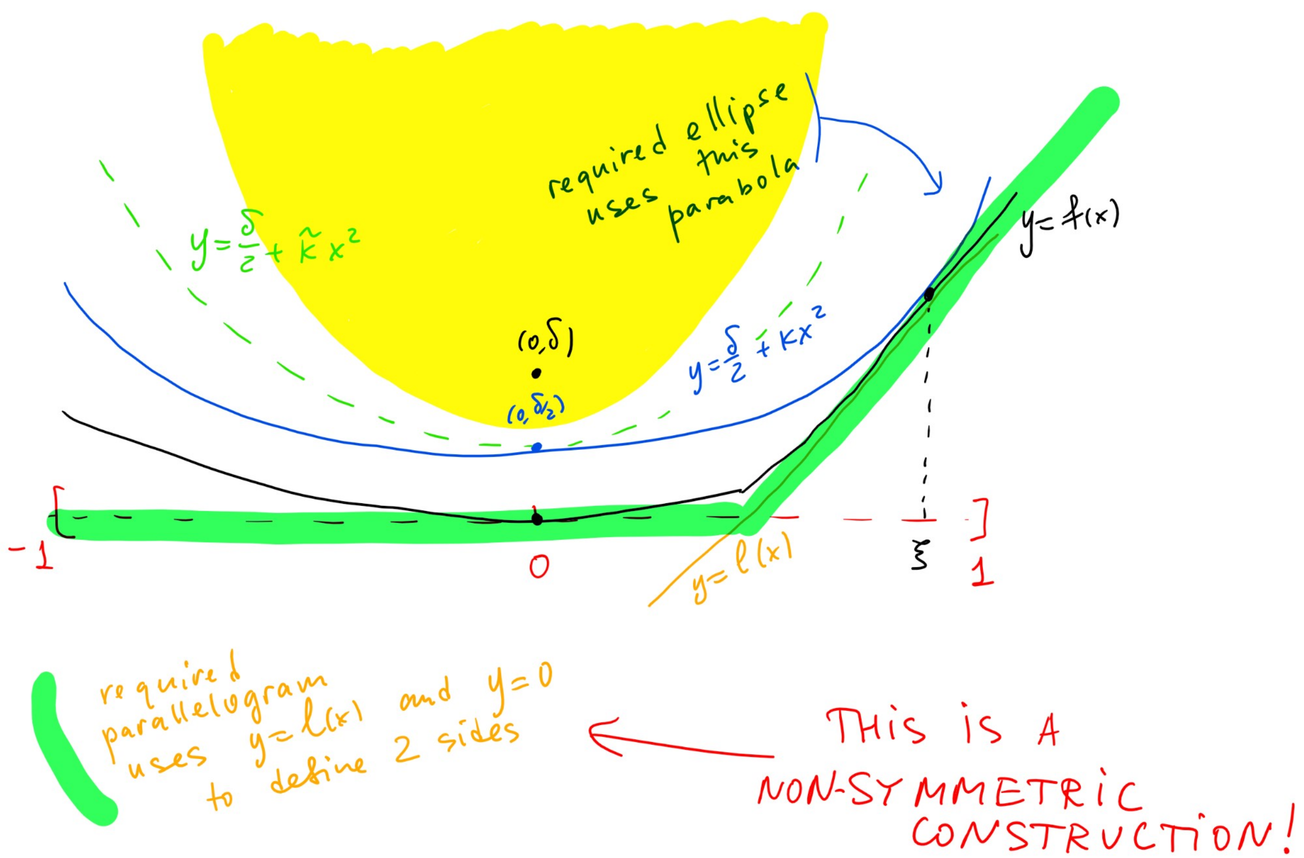
$$f(x) \le \frac{\delta}{2} + kx^2 \quad \textit{for all} \quad x \in [-1,1], \tag{2.13}$$

$$\ell(\xi) = f(\xi), \quad \ell'(\xi) = f'_-(\xi) \ \textit{or} \ \ell'(\xi) = f'_+(\xi), \quad \textit{and} \tag{2.14}$$

$$\frac{\sqrt{\delta + \beta}}{|\alpha|} < \frac{1}{\sqrt{k}}. \tag{2.15}$$

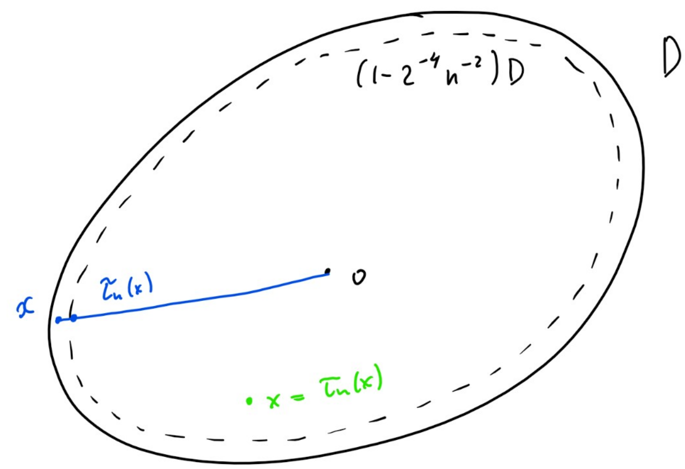


$$\frac{\sqrt{\delta+\beta}}{\alpha} = \frac{\sqrt{\xi f'_-(\xi) - k\xi^2 + \frac{\delta}{2}}}{f'_-(\xi)} < \frac{\sqrt{2(\xi f'_-(\xi) - k\xi^2)}}{f'_-(\xi)} < \frac{\sqrt{2\xi f'_-(\xi)}}{f'_-(\xi)} = \sqrt{\frac{2\xi}{f'_-(\xi)}} \le \frac{1}{\sqrt{k}}$$



Theorem 3.1. *Suppose D is a convex compact set satisfying $B \subset D \subset 2B$. For any $n \geq 1$ and any $\boldsymbol{x} \in D$ define $\tau_n(\boldsymbol{x}) := \boldsymbol{x}$ if $\boldsymbol{x} \in (1 - 2^{-4}n^{-2})D$, and $\tau_n(\boldsymbol{x}) := t\boldsymbol{x}$ where $t > 0$ is the largest scalar satisfying $t\boldsymbol{x} \in (1 - 2^{-4}n^{-2})D$. Then*

$$\lambda_n(\boldsymbol{x}, D) \approx n^{-2} L(\tau_n(\boldsymbol{x}), D) \approx n^{-2} U(\tau_n(\boldsymbol{x}), D). \tag{3.1}$$



Lemma 3.4 ([P2, Proposition 1.4]). *If D is a planar convex body with $\mathbf{0} \in D$, then for any $\boldsymbol{x} \in D$*

$$\lambda_n(\boldsymbol{x}, D) \approx \lambda_n(\mu\boldsymbol{x}, D), \quad \mu \in [1 - 2^{-4}n^{-2}, 1].$$

Corollary 3.5. *For any planar convex domain D, $\boldsymbol{x} \in D$ and $n \geq 1$*

$$\lambda_{2n}(\boldsymbol{x}, D) \approx \lambda_n(\boldsymbol{x}, D). \tag{3.2}$$

Proof. We can invoke the considerations of Remark 3.2 to assume $B \subset D \subset 2B$, so that Theorem 3.1 is applicable. If $\tau_n(\boldsymbol{x}) = \boldsymbol{x}$, then also $\tau_{2n}(\boldsymbol{x}) = \boldsymbol{x}$, so (3.2) follows directly from (3.1). Otherwise, we have $\lambda_n(\tau_n(\boldsymbol{x}), D) \approx \lambda_n(\boldsymbol{x}, D)$ by Lemma 3.4. It is easy to observe that there exists a positive integer m independent of n satisfying

$$(1 - 2^{-4}(2n)^{-2})^m < 1 - 2^{-4}n^{-2}.$$

Therefore, iterating Lemma 3.4 at most m times, we obtain $\lambda_{2n}(\tau_n(\boldsymbol{x}), D) \approx \lambda_{2n}(\boldsymbol{x}, D)$, and (3.2) for $\boldsymbol{x}$ follows from already established (3.2) for $\tau_n(\boldsymbol{x})$. $\square$

Theorem 3.1. *Suppose D is a convex compact set satisfying $B \subset D \subset 2B$. For any $n \geq 1$ and any $\boldsymbol{x} \in D$ define $\tau_n(\boldsymbol{x}) := \boldsymbol{x}$ if $\boldsymbol{x} \in (1 - 2^{-4}n^{-2})D$, and $\tau_n(\boldsymbol{x}) := t\boldsymbol{x}$ where $t > 0$ is the largest scalar satisfying $t\boldsymbol{x} \in (1 - 2^{-4}n^{-2})D$. Then*

$$\lambda_n(\boldsymbol{x}, D) \approx n^{-2}L(\tau_n(\boldsymbol{x}), D) \approx n^{-2}U(\tau_n(\boldsymbol{x}), D). \tag{3.1}$$

$$\lambda_n(\boldsymbol{z}, B) \approx n^{-2}(1 - \|\boldsymbol{z}\|)^{1/2}, \quad \boldsymbol{z} \in (1 - 2^{-7}n^{-2})B. \tag{3.7}$$

lower bound uses Bernstein's inequality for trig pol-s.

$$\lambda_n(\boldsymbol{z}, S) \leq cn^{-2}\sqrt{(\boldsymbol{z})_1(\boldsymbol{z})_2}, \quad \text{for any} \quad \boldsymbol{z} \in [2^{-7}n^{-2}, \tfrac{1}{2}]^2. \tag{3.11}$$

tensor product of polynomials obtained from Chebyshev pol-s divided by one of the zeroes

Near-optimal polynomial in (1.2) can be constructed.

$$\lambda_n(\boldsymbol{x}, D) = \min_{f \in \Pi_{n,d},\, |f(\boldsymbol{x})|=1} \|f\|^2_{L^2(D)}, \quad \boldsymbol{x} \in D. \tag{1.2}$$

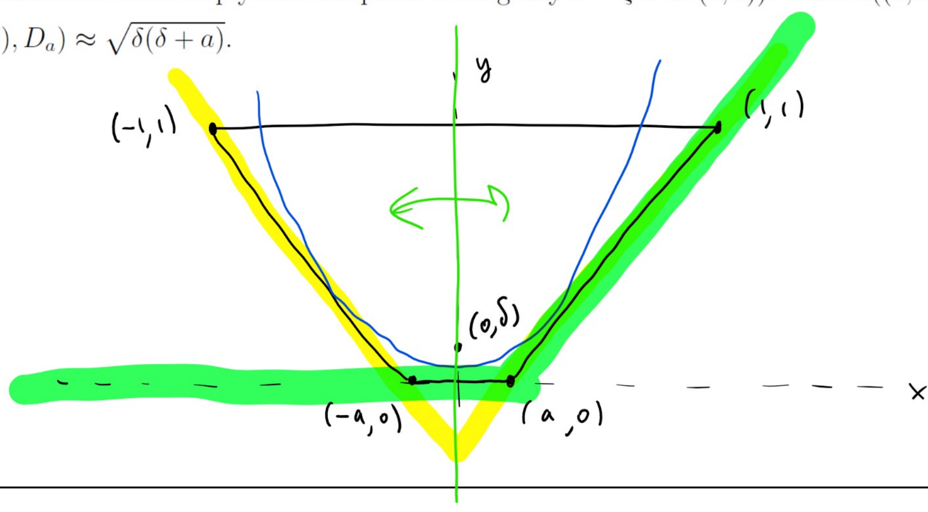
Example 3.9. Let D_a be the trapezoid with the vertices $(\pm a, 0)$, $(\pm 1, 1)$, where $a \in (0, \frac{1}{3}]$. Then for an absolute constant $c > 0$

$$\lambda_n((0,\delta), D_a) \approx n^{-2}\sqrt{\delta(a+\delta)}, \quad \text{for} \quad \delta \in [cn^{-2}, \tfrac{1}{2}]. \tag{3.12}$$

Proof. Let us only provide the main computation and omit other technical details. We follow the proof of Theorem 2.1 and find k as in the proof of Lemma 2.4, which requires the smallest $k > 0$ such that

$$\frac{x-a}{1-a} \leq \frac{\delta}{2} + kx^2 \quad \text{for all} \quad x \in [-1, 1].$$

Then the parabola $y = \frac{\delta}{2} + kx^2$ is tangent to the line $y = \frac{x-a}{1-a}$ and one finds $k \approx (\delta + a)^{-1/2}$ (the restrictions on a and δ imply that the point of tangency $x = \xi$ is in $(0,1)$). Thus $L((0,\delta), D_a) \approx U((0,\delta), D_a) \approx \sqrt{\delta(\delta + a)}$. □

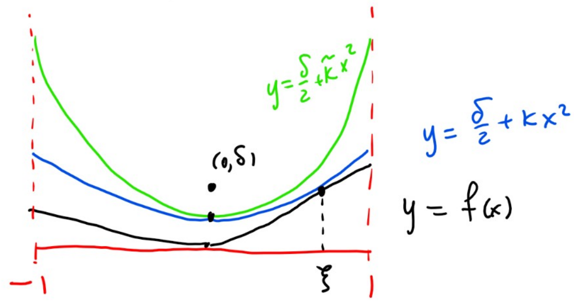


$$(2.1) \qquad L(\boldsymbol{x}, D) := \sup\{(1 - \|\mathcal{L}^{-1}\boldsymbol{x}\|)^{1/2} |\det \mathcal{L}| : \mathcal{L} \in \mathbb{A},\ \boldsymbol{x} \in \mathcal{L}B \subset D\}$$

$$(2.2) \qquad U(\boldsymbol{x}, D) := \inf\{((\mathcal{U}^{-1}\boldsymbol{x})_1(\mathcal{U}^{-1}\boldsymbol{x})_2)^{1/2} |\det \mathcal{U}| : \mathcal{U} \in \mathbb{A},\ \boldsymbol{x} \in \mathcal{U}(\tfrac{1}{2}S),\ D \subset \mathcal{U}S\}.$$

Remark 3.8. It is easy to extend the definitions (2.1) and (2.2) to the higher dimensions, and we conjecture that the corresponding generalizations of Theorems 2.1 and 3.1 are true. While Lemma 2.2 is not hard to generalize, Lemma 2.4 is for two dimensions only. One can observe that in the planar case ($d = 2$) there is only one parameter (k) to define the needed parabola (see (2.16)), while for $d > 2$ there will be $d-1$ parameters which makes generalization of (2.16) and handling the resulting points of tangency much more difficult.

$$(2.16) \qquad k := \inf\left\{\tilde{k} > 0 : \frac{\delta}{2} + \tilde{k}x^2 \geq f(x),\ x \in [-1, 1]\right\}$$



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Corollary 3.5. *For any planar convex domain D, $\boldsymbol{x} \in D$ and $n \geq 1$*

$$(3.2) \qquad \lambda_{2n}(\boldsymbol{x}, D) \approx \lambda_n(\boldsymbol{x}, D).$$

Remark 4.3. Our proof of Corollary 3.5 from Theorem 3.1 readily transfers to higher dimensions. Therefore, generalization of Theorem 3.1 to higher dimensions (see Remark 3.8) would imply existence of optimal polynomial meshes for arbitrary convex bodies, i.e., would confirm Kroo's conjecture for $d > 2$. However, it might be a more accessible task to generalize only Corollary 3.5 which is a much weaker statement than Theorem 3.1.

Remark 4.4. We would also like to comment about similarities and differences of the proofs of existence of optimal polynomial meshes in arbitrary planar convex bodies from this work and from [K4]. A very important part of both proofs is consideration of certain parabolas inside the domain. In our proof we were able to "localize" the problem and work with a fixed interior point; "global" part of the argument was delegated to Tchakaloff's theorem and Lemma 4.1. In [K4], a maximal function was used to prove a "global" tangential Bernstein inequality. While smoothing of the boundary was needed in [K4], we managed to avoid this due to Lemma 3.4.

The rate of growth of $\sup_{x \in D} (\lambda_n(x, D))^{-1}$ is determining for Nikol'skii-type inequalities on D, see:

[DP] Z. Ditzian and A. Prymak, *On Nikol'skii inequalities for domains in $\mathbb{R}^d$*, Constr. Approx. **44** (2016), no. 1, 23–51.

In turn, Nikol'skii-type inequalities appear as a condition in various discretization problems, see:

[DPTT] F. Dai, A. Prymak, V. N. Temlyakov, and S. Yu. Tikhonov, *Integral norm discretization and related problems*, Uspekhi Mat. Nauk **74** (2019), no. 4(448), 3–58 (Russian, with Russian summary); English transl., Russian Math. Surveys **74** (2019), no. 4, 579–630.

Condition C. Let $X_N := \operatorname{span}(u_1, \dots, u_N)$. There exist two constants K_3 and K_4 such that the following Nikol'skii-type inequality holds for all $f \in X_N$:

$$\|f\|_\infty \leqslant K_3 N^{K_4/p} \|f\|_p, \qquad p \in [2, \infty).$$

$$\lambda_n(\boldsymbol{x}, D) := \left(\sum_{j=1}^{N} p_j(\boldsymbol{x})^2 \right)^{-1}$$

Condition E. There exists a constant t such that

$$w(x) := \sum_{i=1}^{N} u_i(x)^2 \leqslant N t^2, \qquad x \in \Omega.$$

Thank you!