Sampling discretization problem for L^1 norm

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Sampling discretization problem

- $C > c > 0$ — fixed
- $L \subset L^p := L^p(\Omega, \mu) \cap C(\Omega)$ — N-dimensional

Main question:

For what $m \in \mathbb{N}$ there are $X_1, \dots, X_m \in \Omega$ such that

$$c\|f\|_p^p \le \frac{1}{m} \sum_{j=1}^{m} |f(X_j)|^p \le C\|f\|_p^p \quad \forall f \in L?$$

Here $\|f\|_p := \left(\int_\Omega |f|^p \, d\mu\right)^{1/p}.$

Comments

- $m \geq N \Rightarrow$ we are interested in the conditions on L under which m is close to N

- **Sampling discretization with weights:** For what $m \in \mathbb{N}$ there are $X_1, \dots, X_m \in \Omega$ and numbers $\lambda_1, \dots, \lambda_m$ such that

$$c\|f\|_p^p \leq \sum_{j=1}^{m} \lambda_j |f(X_j)|^p \leq C\|f\|_p^p \quad \forall f \in L?$$

- Special case: $C = 1 + \varepsilon$, $c = 1 - \varepsilon$, $\varepsilon \in (0, 1)$

Nikolskii-type inequality assumption

Definition. Subspace L satisfies (∞, q) Nikolskii-type inequality assumption (with constant $M > 0$) if

$$\sup_{x\in\Omega} |f(x)| = \|f\|_\infty \leq MN^{1/q}\|f\|_q \quad \forall f \in L.$$

For $q = 2$: $\{u_1, \ldots, u_N\}$ — orthonormal basis in L

$$\sup_{x\in\Omega}\left(|u_1(x)|^2 + \ldots + |u_N(x)|^2\right)^{1/2} \leq M\sqrt{N}.$$

Case $p = 2$

Theorem (V. Temlyakov, I. Limonova, 20).
$\exists\, C_1, C_2, C_3 > 0$: $\forall$ N-dimensional subspace L, satisfying $(\infty, 2)$ Nikolskii-type inequality assumption, $\exists\, m \leq C_1 M^2 N$ points $X_1, \dots, X_m$:

$$C_2\|f\|_2^2 \leq \frac{1}{m}\sum_{j=1}^{m}|f(X_j)|^2 \leq C_3 M^2\|f\|_2^2 \quad \forall f \in L.$$

Case $p \in (2, +\infty)$

Theorem (F. Dai, A. Prymak, A. Shadrin, V. Temlyakov, S. Tikhonov, 20).
Let $p \in [1, \infty)$ and assume that
$e_k(B_p(L), \|\cdot\|_\infty) \leq MN^{1/p}2^{-k/p} \quad 0 \leq k \leq \log N$.
Then $\forall \varepsilon \in (0,1)$ $\exists m \leq CN[\log N]^2$ points such that

$$(1-\varepsilon)\|f\|_p^p \leq \frac{1}{m}\sum_{j=1}^{m}|f(X_j)|^p \leq (1+\varepsilon)\|f\|_p^p \quad \forall f \in L.$$

Theorem (E.K., 20). $p \geq 2 \Rightarrow \forall \varepsilon \in (0,1)$ and $\forall$ N-dimensional subspace L, satisfying (∞, p) Nikolskii-type inequality assumption,
$\exists\, m \leq CN[\log N]^p$ points $X_1, \dots, X_m$:

$$(1-\varepsilon)\|f\|_p^p \leq \frac{1}{m}\sum_{j=1}^{m}|f(X_j)|^p \leq (1+\varepsilon)\|f\|_p^p \quad \forall f \in L.$$

Case $p \in (1,2)$

Theorem (F. Dai, A. Prymak, A. Shadrin, V. Temlyakov, S. Tikhonov, 20). $p \in [1,2) \Rightarrow \forall \varepsilon \in (0,1)$ and $\forall$ N-dimensional subspace L, satisfying $(\infty, 2)$ Nikolskii-type inequality assumption, $\exists\, m \le CN[\log N]^3$ points $X_1, \dots, X_m$:

$$(1-\varepsilon)\|f\|_p^p \le \frac{1}{m}\sum_{j=1}^{m} |f(X_j)|^p \le (1+\varepsilon)\|f\|_p^p \quad \forall f \in L.$$

Theorem (E.K., 20). $p \in (1,2) \Rightarrow \forall \varepsilon \in (0,1)$ and $\forall$ N-dimensional subspace L, satisfying $(\infty, 2)$ Nikolskii-type inequality assumption, $\exists\, m \le CN[\log N]^2$ points $X_1, \dots, X_m$:

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Sharper bound in L^1 case

Theorem (E.K., 21). $\exists C > 0 :\ \forall$ N-dimensional subspace L, satisfying $(\infty, 2)$ Nikolskii-type inequality assumption, $\exists m \leq CM^2 N \log N$ points $X_1, \ldots, X_m$ such that

$$e^{-21}\|f\|_1 \leq \frac{1}{m}\sum_{j=1}^{m}|f(X_j)| \leq e^{21}M^2\|f\|_1 \quad \forall f \in L.$$

Talagrand's theorem

Definition. For a subspace $L \subset L^1(0,1)$ let

$$N(L,\varepsilon) := \min\{m \colon \exists L' \subset \ell_1^m \colon d(L,L') \leq 1+\varepsilon\},$$

where the Banach–Mazur distance

$$d(L,L') = \inf\{\|T\| \cdot \|T^{-1}\| \colon T \colon L \to L'\}.$$

Theorem (M. Talagrand, 90).
L — N-dimensional subspace of $L^1(0,1)$, then $N(L,\varepsilon) \leq C\varepsilon^{-2} N \log N$.

Talagrand's result in terms of sampling discretization with weights

Theorem (reformulation). $L \subset C(\Omega)$ — N-dimensional $\Rightarrow \exists\ m \leq C(\varepsilon) N \log N$ points $X_1, \dots, X_m$ and positive weights $\lambda_1, \dots, \lambda_m$ such that

$$(1-\varepsilon)\|f\|_1 \leq \sum_{j=1}^{m} \lambda_j |f(X_j)| \leq (1+\varepsilon)\|f\|_1 \quad \forall f \in L.$$

Sketch of Talagrand's proof: first steps

1) $L \subset \ell_1^m = L^1(X, \mu)$ with big enough m, $X = \{X_1, \dots, X_m\}$;

2) «Improve» the measure (Lewis change of density): new measure $\mu'(X_j) = \mu'_j$, $L' := \{f' \colon f'(X_j) = \mu_j'^{-1} f(X_j), f \in L\}$ such that $\|f'\|_{L^\infty(X,\mu')} \leq \sqrt{N} \|f'\|_{L^2(X,\mu')}$ $\forall f' \in L'$;

3) Splitting of «big» atoms ($X' = \{X'_1, \dots, X'_{m'}\}$, $m' \leq 3m/2$)

Sketch of Talagrand's proof: the main step

Definition. $(L, \|\cdot\|)$ — Banach space. Let $K\big(L, \|\cdot\|\big)$ be the best constant C such that

$$\mathbb{E}_\varepsilon \left\| \sum_{j=1}^k \varepsilon_j \mathbb{E}_\varepsilon[f(\varepsilon)\varepsilon_j] \right\|^2 \leq C^2 \mathbb{E}_\varepsilon \|f(\varepsilon)\|^2$$

$\forall k \in \mathbb{N}$ and $\forall f : \{-1, 1\}^k \to L$.

Key lemma (Talagrand). Let $X = \{X_1, \ldots, X_m\}$, $\mu(X_j) = \mu_j$, $L \subset L^1(X, \mu)$. Assume, that $\|f\|_{L^\infty(X,\mu)} \leq \theta \|f\|_{L^2(X,\mu)}$ $\forall f \in L$. Then

$$\mathbb{E}_\varepsilon \sup_{f \in B_1(L)} \left| \sum_{j=1}^m \varepsilon_j \mu_j |f(X_j)| \right| \leq 2\sqrt{\pi}\theta K\big(L, \|\cdot\|_{L^1(X,\mu)}\big).$$

Here $\|f\|_{L^p(X,\mu)} = \left(\sum_{j=1}^m \mu_j |f(X_j)|^p \right)^{1/p}.$

Sketch of Talagrand's proof: final steps

4) Key lemma $\Rightarrow$ signs $\{\varepsilon_1^0, \dots, \varepsilon_{m'}^0\}$ such that
$$\Big|\sum_{j=1}^{m'} \varepsilon_j^0 \mu_j' |f'(X_j')|\Big| \le C\sqrt{N} K(L', \|\cdot\|_{L^1(X',\mu')}) \|f'\|_{L^1(X',\mu')};$$

5) Splitting: $X' = Z_1 \sqcup Z_2$:
$$(1-\varepsilon_{m,N})\|f\|_{L^1(X,\mu)} \le \|f\|_{L^1(Z_i,\nu^i)} \le (1+\varepsilon_{m,N})\|f\|_{L^1(X,\mu)},$$
$$\varepsilon_{m,N} = 2C\sqrt{N} K\big(L', \|\cdot\|_{L^1(X',\mu')}\big);$$

6) Choose Z_i: $|Z_i| \le 2^{-1} m' \le 3m/4$ and iterate;

7) L — N-dimensional subspace of $L^1(0,1)$, then
$$N(L,\varepsilon) \le C(\varepsilon)\big[K(L, \|\cdot\|_{L^1})\big]^2 N;$$

8) L — N-dimensional subspace of $L^1(0,1) \Rightarrow$
$$K\big(L, \|\cdot\|_1\big) \le C\sqrt{\log N}.$$

Discretization with equal weights: control of L^1 and L^2 norms

1) L is on discrete $X = \{X_1, \dots, X_m\}$ with big enough m: L^1 and L^2 norms are well discretized;

2) Key lemma + Rudelson's result:

Theorem (M. Rudelson, 99). Let $X = \{X_1, \dots, X_m\}$, $\mu(X_j) = \mu_j$, $L \subset L^2(X, \mu)$ — N-dimensional. Assume, that $\|f\|_{L^\infty(X,\mu)} \le \theta \|f\|_{L^2(X,\mu)}$ $\forall f \in L$. Then

$$\mathbb{E}_\varepsilon \sup_{f \in B_2(L)} \left| \sum_{j=1}^m \varepsilon_j \mu_j |f(X_j)|^2 \right| \le C\theta \sqrt{\log N} \sup_{h \in B_2(L)} \left(\sum_{j=1}^m |h(X_j)|^2 \right)^{1/2}.$$

Discretization with equal weights: good splitting

3) $\exists$ signs $\{\varepsilon_1^0, \dots, \varepsilon_m^0\}$ such that

$$\Big|\sum_{j=1}^m \varepsilon_j^0 |f(X_j)|^2\Big| \le C(m) M\sqrt{N}\sqrt{\log N}\|f\|_{L^2(X)} \text{ and}$$

$$\Big|\sum_{j=1}^m \varepsilon_j^0 |f(X_j)|\Big| \le C(m) M\sqrt{N} K\big(L, \|\cdot\|_{L^1(X)}\big)\|f\|_{L^1(X)}$$

4) $\exists Z \subset X: \ |Z| \le 2^{-1}|X|$,

$$(\tfrac{1}{2} - A)\|f\|_{L^1(X)} \le \|f\|_{L^1(Z)} \le (\tfrac{1}{2} + A)\|f\|_{L^1(X)} \text{ and}$$

$$(\tfrac{1}{2} - A)\|f\|^2_{L^2(X)} \le \|f\|^2_{L^2(Z)} \le (\tfrac{1}{2} + A)\|f\|^2_{L^2(X)}$$

$$A = C(m) M\sqrt{N} \max\{\sqrt{\log N}, K\big(L, \|\cdot\|_{L^1(X)}\big)\}$$

5) iterate the above.

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Thank You!