On sampling discretization in L_2

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joint work with V.N. Temlyakov

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Outline

Let Ω be a nonempty subset of $\mathbb{R}^d$ with the probability measure μ.

Definition

We say that a linear subspace X_N of the $L_q(\Omega)$, $1 \le q < \infty$, admits *the Marcinkiewicz-type discretization theorem* with parameters m, q and constants C_1, C_2 if there exist a set of points $\{\xi^j\}_{j=1}^m \subset \Omega$ such that for any $f \in X_N$ we have

$$C_1\|f\|_q^q \le \frac{1}{m}\sum_{j=1}^{m}|f(\xi^j)|^q \le C_2\|f\|_q^q.$$

Condition E

We say that an orthonormal system $\{u_i(x)\}_{i=1}^N$ defined on Ω satisfies Condition E with a constant $t > 0$ if for all $x \in \Omega$

$$\sum_{i=1}^{N} |u_i(x)|^2 \le Nt^2.$$

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Note that integration of the above inequality over $x \in \Omega$ gives $t \geq 1$.

Theorem (M. Rudelson, 1999)

Let $\Omega_M = \{x^j\}_{j=1}^M$ be a discrete set with the probability measure $\mu_M(x^j) = 1/M$, $j = 1, \dots, M$. Assume that a real orthonormal system $\{u_i(x)\}_{i=1}^N$ satisfies Condition E on Ω_M. Then for every $\epsilon > 0$ there exists a set $J \subset \{1, \dots, M\}$ of indices with cardinality

$$m := |J| \le C\frac{t^2}{\epsilon^2} N \log \frac{Nt^2}{\epsilon^2}$$

such that for any $f = \sum_{i=1}^N c_i u_i$ we have

$$(1-\epsilon)^2 \|f\|_2^2 \le \frac{1}{m}\sum_{j\in J} f(x^j)^2 \le (1+\epsilon)^2\|f\|_2^2.$$

Theorem (V.N. Temlyakov, 2018)

Let $\{u_i(x)\}_{i=1}^N$ be a real orthonormal in $L_2(\Omega, \mu)$ system satisfying Condition E. Then for every $\epsilon > 0$ there exists a set $\{\xi^j\}_{j=1}^m \subset \Omega$ with

$$m \le C\frac{t^2}{\epsilon^2} N \log N$$

such that for any $f = \sum_{i=1}^N c_i u_i$ we have

$$(1-\epsilon)\|f\|_2^2 \le \frac{1}{m}\sum_{j=1}^m f(\xi^j)^2 \le (1+\epsilon)\|f\|_2^2.$$

Theorem (V.N. Temlyakov, 2017)

Let $\Omega_M = \{x^j\}_{j=1}^M$ be a discrete set with the probability measure $\mu_M(x^j) = 1/M$, $j = 1, \dots, M$. Assume that $\{u_i(x)\}_{i=1}^N$ is an orthonormal on Ω_M system (real or complex). Assume in addition that this system has the following property: for all $j = 1, \dots, M$ we have

$$\sum_{i=1}^N |u_i(x^j)|^2 = N.$$

Then there is an absolute constant C_1 such that there exists a subset $J \subset \{1, 2, \dots, M\}$ with the property: $m := |J| \le C_1 N$ and for any $f = \sum_{i=1}^N c_i u_i$ we have

$$C_2 \|f\|_2^2 \le \frac{1}{m} \sum_{j \in J} |f(x^j)|^2 \le C_3 \|f\|_2^2,$$

where C_2 and C_3 are absolute positive constants.

Theorem (V.N. Temlyakov and IL, 2020)

Let $\Omega \subset \mathbb{R}^d$ be a nonempty set with the probability measure μ. Assume that $\{u_i(x)\}_{i=1}^N$ is a real (or complex) orthonormal system in $L_2(\Omega, \mu)$ satisfying Condition E. Then there is an absolute constant C_1 such that there exists a set $\{\xi^j\}_{j=1}^m \subset \Omega$ of $m \leq C_1 t^2 N$ points with the property: for any $f = \sum_{i=1}^N c_i u_i$ we have

$$C_2 \|f\|_2^2 \leq \frac{1}{m} \sum_{j=1}^m |f(\xi^j)|^2 \leq C_3 t^2 \|f\|_2^2,$$

where C_2 and C_3 are absolute positive constants.

Let X_N be an N-dimensional subspace of real (or complex) space of continuous functions $\mathcal{C}(\Omega)$.

Nikol'skii inequality. We say that X_N satisfies the Nikol'skii inequality for the pair $(2, \infty)$ if there exists a constant $t > 0$ such that

$$\|f\|_\infty \leq tN^{\frac{1}{2}}\|f\|_2, \quad \forall f \in X_N. \tag{1}$$

We point out that condition (1) with $X_N = \operatorname{span}(u_1, \dots, u_N)$, where $\{u_j\}_{j=1}^N$ is an orthonormal system, is equivalent to Condition E.

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Proposition

Let X_N be an N-dimensional subspace of $\mathcal{C}(\Omega)$. Then for any orthonormal basis $\{u_i\}_{i=1}^N$ of $X_N \subset L_2(\Omega, \mu)$ we have that for $x \in \Omega$

$$\sup_{f \in X_N, f \neq 0} \frac{|f(x)|}{\|f\|_2} = \left(\sum_{i=1}^N |u_i(x)|^2\right)^{1/2}.$$

Theorem (V.N. Temlyakov and IL, 2020)

Let $\Omega \subset \mathbb{R}^d$ be a nonempty compact set with the probability measure μ. Assume that $X_N \subset \mathcal{C}(\Omega)$ satisfies the Nikol'skii inequality (1). Then there is an absolute constant C_1' such that there exists a set $\{\xi^j\}_{j=1}^m \subset \Omega$ of $m \le C_1' t^2 N$ points with the property: for any $f \in X_N$ we have

$$C_2' \|f\|_2^2 \le \frac{1}{m} \sum_{j=1}^{m} |f(\xi^j)|^2 \le C_3' t^2 \|f\|_2^2,$$

where C_2' and C_3' are absolute positive constants.

Outline

$$\frac{1}{m}\sum_{j=1}^{m}|f(\xi^j)|^2 \to \sum_{j=1}^{m}\lambda_j|f(\xi^j)|^2$$

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Theorem (J. Batson, D.A. Spielman, and N. Srivastava, 2014)

Let $\Omega_M = \{x^j\}_{j=1}^M$ be a discrete set with the probability measure $\mu_M(x^j) = 1/M$, $j = 1, \dots, M$, and let X_N be an N-dimensional subspace of real functions defined on Ω_M. Then for any number $b > 1$ there exists a set of weights $\lambda_j \geq 0$ such that $|\{j : \lambda_j \neq 0\}| \leq \lceil bN \rceil$ so that for any $f \in X_N$ we have

$$\|f\|_2^2 \leq \sum_{j=1}^{M}\lambda_j f(x^j)^2 \leq \frac{b+1+2\sqrt{b}}{b+1-2\sqrt{b}}\|f\|_2^2.$$

As observed in [DPTT, 2019], this last theorem with a general probability space (Ω, μ) in place of the discrete space (Ω_M, μ_M) remains true if $X_N \subset L_4(\Omega, \mu)$.

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Theorem (F.Dai, A.Prymak, A.Shadrin, V.Temlyakov, S.Tikhonov, 2020)

If X_N is an N-dimensional subspace of the real $L_2(\Omega, \mu)$, then for any $b \in (1, 2]$, there exist a set of $m \le \lceil bN \rceil$ points $\xi^1, \dots, \xi^m \in \Omega$ and a set of nonnegative weights λ_j, $j = 1, \dots, m$, such that

$$\|f\|_2^2 \le \sum_{j=1}^{m} \lambda_j f(\xi^j)^2 \le \frac{C}{(b-1)^2} \|f\|_2^2, \quad \forall f \in X_N,$$

where $C > 1$ is an absolute constant.

Theorem (V.N. Temlyakov and IL, 2020)

If X_N is an N-dimensional subspace of the complex $L_2(\Omega, \mu)$, then there exist three absolute positive constants C_1', c_0', C_0', a set of $m \leq C_1' N$ points $\xi^1, \dots, \xi^m \in \Omega$, and a set of nonnegative weights λ_j, $j = 1, \dots, m$, such that

$$c_0' \|f\|_2^2 \leq \sum_{j=1}^{m} \lambda_j |f(\xi^j)|^2 \leq C_0' \|f\|_2^2, \quad \forall f \in X_N.$$

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Remark

This Theorem holds with $m \le \lceil 2bN \rceil$, $b \in (1,2]$, and $c_0' = 1$, $C_0' = C(b-1)^{-2}$, where C is an absolute constant from Theorem [DPSTT].

Outline

Theorem (A. Marcus, D.A. Spielman, and N. Srivastava, 2015)

Let a system of vectors $\mathbf{v}_1, \dots, \mathbf{v}_M$ *from* $\mathbb{C}^N$ *have the following properties: for all* $\mathbf{w} \in \mathbb{C}^N$

$$\sum_{j=1}^{M} |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 = \|\mathbf{w}\|_2^2 \qquad (2)$$

and for some $\epsilon > 0$

$$\|\mathbf{v}_j\|_2^2 \le \epsilon, \qquad j = 1, \dots, M.$$

Then there is a partition of $\{1, 2, \dots, M\}$ *into two sets* S_1 *and* S_2 *such that for all* $\mathbf{w} \in \mathbb{C}^N$ *and for each* $i = 1, 2$

$$\sum_{j \in S_i} |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 \le \frac{(1 + \sqrt{2\epsilon})^2}{2} \|\mathbf{w}\|_2^2.$$

Lemma (S. Nitzan, A. Olevskii, and A. Ulanovskii, 2016)

Let a system of vectors $\mathbf{v}_1, \dots, \mathbf{v}_M$ *from* $\mathbb{C}^N$ *satisfy* (2) *for all* $\mathbf{w} \in \mathbb{C}^N$ *and*

$$\|\mathbf{v}_j\|_2^2 = N/M, \qquad j = 1, \dots, M.$$

Then there is a subset $J \subset \{1, 2, \dots, M\}$ *such that for all* $\mathbf{w} \in \mathbb{C}^N$

$$c_0 \|\mathbf{w}\|_2^2 \le \frac{M}{N} \sum_{j \in J} |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 \le C_0 \|\mathbf{w}\|_2^2,$$

where c_0 *and* C_0 *are some absolute positive constants.*

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where c_0 *and* C_0 *are some absolute positive constants.*

Remark (V.N. Temlyakov, 2017)

For the cardinality of the subset J *from the previous lemma we have*

$$c_0 N \le |J| \le C_0 N.$$

Lemma

Let a system of vectors $\mathbf{v}_1, \dots, \mathbf{v}_M$ *from* $\mathbb{C}^N$ *satisfy* (2) *for all* $\mathbf{w} \in \mathbb{C}^N$ *and*

$$\|\mathbf{v}_j\|_2^2 \le \theta N/M, \qquad \theta \le M/N, \qquad j = 1, \dots, M.$$

Then there is a subset $J \subset \{1, 2, \dots, M\}$ *such that for all* $\mathbf{w} \in \mathbb{C}^N$

$$c_0 \theta \|\mathbf{w}\|_2^2 \le \frac{M}{N} \sum_{j \in J} |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 \le C_0 \theta \|\mathbf{w}\|_2^2, \quad |J| \le C_1 \theta N,$$

where c_0, C_0, *and* C_1 *are some absolute positive constants.*

Corollary

Let a system of vectors $\mathbf{v}_1, \dots, \mathbf{v}_M$ *from* $\mathbb{C}^N$ *satisfy* (2) *for all* $\mathbf{w} \in \mathbb{C}^N$*. Then there exists a set of weights* $\lambda_j \geq 0$, $j = 1, \dots, M$, *such that* $|\{j : \lambda_j \neq 0\}| \leq 2C_1 N$ *and for all* $\mathbf{w} \in \mathbb{C}^N$ *we have*

$$c_0 \|\mathbf{w}\|_2^2 \leq \sum_{j=1}^{M} \lambda_j |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 \leq C_0 \|\mathbf{w}\|_2^2.$$

where c_0, C_0, *and* C_1 *are absolute positive constants from Lemma.*

Sketch of proof.

Assume that $\|\mathbf{v}_1\|_2 = \min_{j=1,\dots,M} \|\mathbf{v}_j\|_2$. Let $n_1, \dots, n_M$ be natural numbers such that for every j, $1 \le j \le M$,

$$\|\mathbf{v}_1\|_2^2 \le \frac{\|\mathbf{v}_j\|_2^2}{n_j} < 2\|\mathbf{v}_1\|_2^2.$$

Denote

$$M' = \sum_{j=1}^{M} n_j.$$

We build a system V of vectors $\mathbf{v}'_1, \dots, \mathbf{v}'_{M'}$ from $\mathbb{C}^N$ in the following way: for every j, $1 \le j \le M$, we include in V n_j copies of the vector $\mathbf{v}_j/\sqrt{n_j}$.

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Theorem (V.N. Temlyakov and IL, 2020)

Let $\Omega_M = \{x^j\}_{j=1}^M$ *be a discrete set with the probability measure* $\mu_M(x^j) = 1/M$, $j = 1, \dots, M$. *Assume that* $\{u_i(x)\}_{i=1}^N$ *is an orthonormal on* Ω_M *system (real or complex). Then there is an absolute constant* C_1 *such that there exists a set of weights* $\lambda_j \geq 0$, $j = 1, \dots, M$, *with the property:* $m := |\{j : \lambda_j \neq 0\}| \leq C_1 N$ *and for any* $f = \sum_{i=1}^N c_i u_i$ *we have*

$$c_0\|f\|_2^2 \leq \sum_{j=1}^M \lambda_j |f(x^j)|^2 \leq C_0\|f\|_2^2,$$

where c_0 *and* C_0 *are from Lemma.*

Lemma

Let a system of vectors $\mathbf{v}_1, \dots, \mathbf{v}_M$ *from* $\mathbb{C}^N$ *satisfy* (2) *for all* $\mathbf{w} \in \mathbb{C}^N$ *and*

$$\|\mathbf{v}_j\|_2^2 \le \theta N/M, \qquad \theta \le M/N, \qquad j = 1, \dots, M.$$

Then there is a subset $J \subset \{1, 2, \dots, M\}$ *such that for all* $\mathbf{w} \in \mathbb{C}^N$

$$c_0 \theta \|\mathbf{w}\|_2^2 \le \frac{M}{N} \sum_{j \in J} |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 \le C_0 \theta \|\mathbf{w}\|_2^2, \quad |J| \le C_1 \theta N,$$

where c_0, C_0, *and* C_1 *are some absolute positive constants.*

Proposition (N.J. Harvey and N. Olver, 2014,
S. Nitzan, A. Olevskii, and A. Ulanovskii, 2016)

Let $\mathbf{v}_1, \ldots, \mathbf{v}_M \in \mathbb{C}^N$ *and* $\delta > 0$ *be such that* $\|\mathbf{v}_j\|_2^2 \le \delta$ *for all* $j = 1, \ldots, M$. *If*

$$\alpha \|\mathbf{w}\|_2^2 \le \sum_{j=1}^{M} |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 \le \beta \|\mathbf{w}\|_2^2, \qquad \forall \mathbf{w} \in \mathbb{C}^N,$$

with some numbers $\beta \ge \alpha > \delta$, *then there exists a partition of* $\{1, \ldots, M\}$ *into* S_1 *and* S_2 *such that for each* $i = 1, 2$:

$$\frac{1 - 5\sqrt{\delta/\alpha}}{2} \alpha \|\mathbf{w}\|_2^2 \le \sum_{j \in S_i} |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 \le \frac{1 + 5\sqrt{\delta/\alpha}}{2} \beta \|\mathbf{w}\|_2^2, \quad \forall \mathbf{w} \in \mathbb{C}^N.$$

Lemma (S. Nitzan, A. Olevskii, and A. Ulanovskii, 2016)

Let $0 < \delta < 1/100$, *and let* $\alpha_j, \beta_j, j = 0, 1, \dots,$ *be defined inductively*

$$\alpha_0 = \beta_0 = 1, \quad \alpha_{j+1} := \alpha_j \frac{1 - 5\sqrt{\delta/\alpha_j}}{2}, \quad \beta_{j+1} := \beta_j \frac{1 + 5\sqrt{\delta/\alpha_j}}{2}.$$

Then there exist a positive absolute constant C *and a number* $L \in \mathbb{N}$ *such that*

$$\alpha_j \geq 100\delta, \quad j \leq L, \quad 25\delta \leq \alpha_{L+1} < 100\delta, \quad \beta_{L+1} < C\alpha_{L+1}.$$

Theorem (V.N. Temlyakov and IL, 2020)

Let $\Omega_M = \{x^j\}_{j=1}^M$ be a discrete set with the probability measure $\mu_M(x^j) = 1/M$, $j = 1, \dots, M$. Assume that $\{u_i(x)\}_{i=1}^N$ is an orthonormal on Ω_M system (real or complex). Assume in addition that this system has the following property: for all $j = 1, \dots, M$ and for some $t > 0$ we have

$$\sum_{i=1}^N |u_i(x^j)|^2 \le Nt^2.$$

Then there is an absolute constant C_1 such that there exists a subset $J \subset \{1, 2, \dots, M\}$ with the property: $m := |J| \le C_1 t^2 N$ and for any $f = \sum_{i=1}^N c_i u_i$ we have

$$C_2 \|f\|_2^2 \le \frac{1}{m} \sum_{j \in J} |f(x^j)|^2 \le C_3 t^2 \|f\|_2^2,$$

where C_2 and C_3 are absolute positive constants.

Proposition (F. Dai, A. Prymak, V.N. Temlyakov, S.Yu. Tikhonov, 2019)

Let $Y_N := \operatorname{span}(u_1(x), \dots, u_N(x))$ *with* $\{u_i(x)\}_{i=1}^N$ *being a real (or complex) orthonormal on* Ω *with respect to a probability measure* μ *basis for* Y_N. *Assume that* $\|u_i\|_4 := \|u_i\|_{L_4(\Omega,\mu)} < \infty$ *for all* $i = 1, \dots, N$. *Then for any* $\delta > 0$ *there exists a set* $\Omega_M = \{x^j\}_{j=1}^M \subset \Omega$ *such that for any* $f \in Y_N$

$$\left| \|f\|^2_{L_2(\Omega,\mu)} - \|f\|^2_{L_2(\Omega_M,\mu_M)} \right| \le \delta \|f\|^2_{L_2(\Omega,\mu)},$$

where

$$\|f\|^2_{L_2(\Omega_M,\mu_M)} := \frac{1}{M} \sum_{j=1}^M |f(x^j)|^2.$$

Proposition

Let $X_N = \operatorname{span}(w_1, \dots, w_N)$ *be a subspace of complex* $L_2(\Omega, \mu)$*. Suppose that* $w_j = u_j + iv_j$*, where* u_j*,* v_j *are real functions,* $j = 1, \dots, N$*. Denote* $Y_S := \operatorname{span}(u_1, \dots, u_N, v_1, \dots, v_N)$*,* $S := \dim Y_S \le 2N$*, a real subspace of* $L_2(\Omega, \mu)$*. Then*

$$Y_S \in \mathcal{M}^w(m, 2, C_1, C_2) \quad \textit{implies} \quad X_N \in \mathcal{M}^w(m, 2, C_1, C_2).$$

Moreover, for discretization of X_N *we can use the same points and weights as for discretization of* Y_S*.*

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Moreover, for discretization of X_N *we can use the same points and weights as for discretization of* Y_S*.*

Take an $f \in X_N$ and write $f = f_R + if_I, \quad f_R, f_I \in Y_S$.
$\xi^1, \dots, \xi^m \in \Omega$, λ_ν, $\nu = 1, \dots, m$:

$$C_1\|g\|_2^2 \le \sum_{\nu=1}^{m} \lambda_\nu |g(\xi^\nu)|^2 \le C_2\|g\|_2^2. \text{ Then}$$

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$\xi^1, \dots, \xi^m \in \Omega$, λ_ν, $\nu = 1, \dots, m$:

$$C_1 \|g\|_2^2 \le \sum_{\nu=1}^{m} \lambda_\nu |g(\xi^\nu)|^2 \le C_2 \|g\|_2^2. \text{ Then}$$

$$\sum_{\nu=1}^{m} \lambda_\nu |f(\xi^\nu)|^2 = \sum_{\nu=1}^{m} \lambda_\nu (|f_R(\xi^\nu)|^2 + |f_I(\xi^\nu)|^2) \le C_2(\|f_R\|_2^2 + \|f_I\|_2^2) = C_2 \|f\|_2^2.$$

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Thank you for your attention!