# Sampling recovery of multivariate functions in the uniform norm

Kateryna Pozharska

Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

joint work with Tino Ullrich
Technische Universität Chemnitz, Germany

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$f\colon D\subset\mathbb{R}^d\to\mathbb{C}$

$\mathbf{X}=\{\mathbf{x}^1,\dots,\mathbf{x}^n\}$, $\mathbf{x}^i\subset D$, $i=1,\dots,n$

$f(\mathbf{X})$: $f(\mathbf{x}^1),\dots,f(\mathbf{x}^n)$

**Reconstruct $f$ from samples**

- The sampling nodes should work for a class of functions simultaneously
- $H(K)$ is a reproducing kernel Hilbert space (RKHS) with bounded kernel $K\colon D\times D\to\mathbb{C}$

- Control the worst-case error

$$\sup_{\|f\|_{H(K)}\le 1}\|f-S_{\mathbf{X}}^m f\|_{\ell_\infty(D)}$$

- Discuss the power of standard information in the uniform norm
- Obtain new recovery guarantees for concrete Sobolev type spaces

**Linear information**: Gelfand numbers / widths, approximation numbers / linear widths

$$a_n\left(\mathrm{Id}\colon H(K)\to F\right) := \inf_{\substack{A\in\mathcal{L}(H(K),F)\\ \mathrm{rank}A<n}} \sup_{\|f\|_{H(K)}\le 1} \|f-Af\|_F \tag{1}$$

**Standard information**: sampling numbers

$$g_n\left(\mathrm{Id}\colon H(K)\to F\right) := \inf_{\mathbf{X}=\{\mathbf{x}^1,\ldots,\mathbf{x}^{n-1}\}} \inf_{R\in\mathcal{L}(\mathbb{C}^{n-1},F)} \sup_{\|f\|_{H(K)}\le 1} \|f-R\left(f(\mathbf{X})\right)\|_F \tag{2}$$

$$a_n(\mathrm{Id})\le g_n(\mathrm{Id})$$

D. Krieg and M. Ullrich, L. Kämmerer, T. Ullrich and T. Volkmer, M. Ullrich, M. Moeller and T. Ullrich, N. Nagel, M. Schäfer and T. Ullrich, V.N. Temlyakov

F. Y. Kuo, G. W. Wasilkowski and H. Woźniakowski

$$\sigma_n = \mathcal{O}(n^{-p}) \quad \Longrightarrow \quad q_{2,\varrho_D}^{\mathrm{lin}} = p$$

$$N_{K,\varrho_D}(k) = \mathcal{O}(k),\ \sigma_n = \mathcal{O}(n^{-p}) \quad \Longrightarrow \quad q_\infty^{\mathrm{std}} = q_\infty^{\mathrm{lin}} = p - 1/2$$

If $N_{K,\varrho_D}(k) = \mathcal{O}(k)$ we obtain for $\mathrm{Id}$: $\boldsymbol{H(K) \to \ell_\infty(D)}$

$$g_m(\mathrm{Id}) \le C_{\varrho_D,K} \min\{a_{\lfloor m/(b \log m)\rfloor}(\mathrm{Id})\,,\, \sqrt{\log m} \cdot a_{\lfloor cm \rfloor}(\mathrm{Id})\}$$

$$N_{K,\varrho_D}(k) = \mathcal{O}(k^u),\ u > 1 \quad \Longrightarrow \quad q_\infty^{\mathrm{std}} \ge p - u/2$$

$$\|f\|_{L_2(D,\varrho_D)} = \left(\int_D |f(\boldsymbol{x})|^2 \mathrm{d}\varrho_D(\boldsymbol{x})\right)^{1/2}$$

$$\|f\|_{\ell_\infty(D)} = \sup_{\boldsymbol{x}\in D} |f(\boldsymbol{x})|$$

$$\forall f \in H(K),\ \forall \boldsymbol{x} \in D \qquad f(\boldsymbol{x}) = (f, K(\cdot,\boldsymbol{x}))_{H(K)}$$

$$\|K\|_\infty^2 := \sup_{\boldsymbol{x}\in D} K(\boldsymbol{x},\boldsymbol{x}) < \infty \tag{3}$$

$$\|f\|_{\ell_\infty(D)} \le \|K\|_\infty \cdot \|f\|_{H(K)}$$

$$\operatorname{tr} K := \|K\|_2^2 = \int_D K(\boldsymbol{x},\boldsymbol{x})\mathrm{d}\varrho_D(\boldsymbol{x}) < \infty \tag{4}$$

$$\forall f \in H(K),\ \forall \boldsymbol{x} \in D \qquad f(\boldsymbol{x}) = (f, K(\cdot, \boldsymbol{x}))_{H(K)}$$

$$\mathrm{Id}\colon H(K) \to L_2(D, \varrho_D), \qquad W_{\varrho_D} = \mathrm{Id}^* \circ \mathrm{Id}\colon H(K) \to H(K),$$

where $(\mathrm{Id} f, g)_{L_2(D,\varrho_D)} = (f, \mathrm{Id}^* g)_{H(K)}$.

$(\lambda_n)_{n=1}^{\infty}$ — rearrangement of eigenvalues of $W_{\varrho_D}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$,
$(\sigma_n)_{n=1}^{\infty}$ — set of singular values, i.e., $\sigma_j = \sqrt{\lambda_j}$, $j = 1, 2, \ldots$,

$(e_n^*(\mathbf{x}))_{n=1}^{\infty} \subset H(K)$ — set of right singular functions,
$(\eta_n(\mathbf{x}))_{n=1}^{\infty} = (\sigma_n^{-1} e_n^*(\mathbf{x}))_{n=1}^{\infty} \subset L_2(D, \varrho_D)$.

**Mercer kernel**

$$K(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{\infty} \overline{e_k^*(\mathbf{y})} e_k^*(\mathbf{x}), \qquad \mathbf{x}, \mathbf{y} \in D \tag{5}$$

# Least squares algorithm

**D. Krieg, M. Ullrich** Function values are enough for $L_2$-approximation. *Found. Comp. Math.*, to appear. arXiv:math/1905.02516v3.

**L. Kämmerer, T. Ullrich, T. Volkmer** *Worst case recovery guarantees for least squares approximation using random samples*, arXiv: 1911.10111, 2019.

**Recovery operator** $S_{\mathbf{X}}^m := \sum_{k=1}^{m-1} c_k \eta_k$

$$\mathbf{f} := (f(\mathbf{x}^1), \dots, f(\mathbf{x}^n))^\top, \ \mathbf{c} := (c_1, \dots, c_{m-1})^\top, \ \ (\eta_k(\mathbf{x}))_{k=1}^{\infty} = (\sigma_k^{-1} e_k^*(\mathbf{x}))_{k=1}^{\infty}$$

$$\mathbf{L}_{n,m} := \mathbf{L}_{n,m}(\mathbf{X}) = \begin{pmatrix} \eta_1(\mathbf{x}^1) & \eta_2(\mathbf{x}^1) & \cdots & \eta_{m-1}(\mathbf{x}^1) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_1(\mathbf{x}^n) & \eta_2(\mathbf{x}^n) & \cdots & \eta_{m-1}(\mathbf{x}^n) \end{pmatrix} \tag{6}$$

Solve the over-determined linear system

$$\mathbf{L}_{n,m} \cdot \mathbf{c} = \mathbf{f}$$

via least squares, i.e., compute

$$\mathbf{c} = (\mathbf{L}_{n,m}^* \mathbf{L}_{n,m})^{-1} \mathbf{L}_{n,m}^* \cdot \mathbf{f} \tag{7}$$

# Weighted least squares regression

**Input**: $\mathbf{X} = (\mathbf{x}^1, ..., \mathbf{x}^n) \in D^n$ set of distinct sampling nodes,
$\mathrm{f} = (f(\mathbf{x}^1), ..., f(\mathbf{x}^n))^\top$ samples of $f$ evaluated at the nodes from $\mathbf{X}$,
$m \in \mathbb{N}$ $m < n$ such that the matrix $\widetilde{\mathbf{L}}_{k,m}$ has full (column) rank.

Compute reweighted samples $\boldsymbol{g} := (g_j)_{j=1}^n$ with

$$g_j := \begin{cases} 0, & \varrho_m(\mathbf{x}^j) = 0, \\ f(\mathbf{x}^j)/\sqrt{\varrho_m(\mathbf{x}^j)}, & \varrho_m(\mathbf{x}^j) \neq 0\,. \end{cases}$$

Solve the over-determined linear system $\widetilde{\mathbf{L}}_{k,m} \cdot (\tilde{c}_1, ..., \tilde{c}_{m-1})^\top = \mathrm{g}$ ,

$$\widetilde{\mathbf{L}}_{k,m} := \left(l_{j,k}\right)_{j=1,k=1}^{n,m-1}, \quad l_{j,k} := \begin{cases} 0, & \varrho_m(\mathbf{x}^j) = 0, \\ \eta_k(\mathbf{x}^j)/\sqrt{\varrho_m(\mathbf{x}^j)}, & \varrho_m(\mathbf{x}^j) \neq 0, \end{cases}$$

via least squares, i.e., compute $(\tilde{c}_1, ..., \tilde{c}_{m-1})^\top := (\widetilde{\mathbf{L}}^*_{k,m}\widetilde{\mathbf{L}}_{k,m})^{-1}\, \widetilde{\mathbf{L}}^*_{k,m} \cdot \mathbf{g}$.
**Output**: $\tilde{\mathbf{c}} = (\tilde{c}_1, ..., \tilde{c}_{m-1})^\top \in \mathbb{C}^{m-1}$ coefficients of the approximant

$$\widetilde{S}^m_{\mathbf{X}} f := \sum_{k=1}^{m-1} \tilde{c}_k \eta_k\,.$$

**D. Krieg, M. Ullrich** Function values are enough for $L_2$-approximation. *Found. Comp. Math.*, to appear. arXiv:math/1905.02516v3.

**A. Cohen, G. Migliorati** Optimal weighted least-squares methods. *SMAI J. Comput. Math.*, 3:181–203, 2017.

**V. N. Temlyakov** On optimal recovery in $L_2$. *J. Complexity*, to appear. arXiv: math/2010.03103.

$$\varrho_m(\mathrm{x}) = \frac{1}{2}\Big(\frac{1}{m-1}\sum_{k=1}^{m-1}|\eta_k(\mathrm{x})|^2 + \frac{1}{\sum_{k=m}^{\infty}\lambda_k}\sum_{k=m}^{\infty}|e_k^*(\mathrm{x})|^2\Big) \tag{8}$$

$$\varrho_m'(\mathrm{x}) = \frac{1}{2(m-1)}\sum_{k=1}^{m-1}|\eta_k(\mathrm{x})|^2 + \frac{1}{2} \tag{9}$$

$\mathbf{X} = (\boldsymbol{x}^1, ..., \boldsymbol{x}^n)$ are drawn i.i.d. with respect to $\varrho_m(\cdot)d\varrho_D$

For every $f \in H(K)$ with a Mercer kernel $K$, it holds

$$f(\mathbf{x}) = \sum_{k=1}^{\infty} (f, e_k^*)_{H(K)} e_k^*(\mathbf{x}).$$

Let

$$P_m f := \sum_{k=1}^{m} (f, e_k^*)_{H(K)} e_k^*(\cdot)$$

be the projection onto the space $\mathrm{span}\{e_1^*(\cdot), \dots, e_m^*(\cdot)\}$.

$$\|f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)} \leq \|f - P_{m-1} f\|_{\ell_\infty(D)} + \|P_{m-1} f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)}$$

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$$\|f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)} \le \|f - P_{m-1} f\|_{\ell_\infty(D)} + \|P_{m-1} f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)}$$

$$\sup_{\|f\|_{H(K)} \le 1} \|f - P_{m-1} f\|_{\ell_\infty(D)} \le \sqrt{2 \sum_{k \ge \lfloor m/2 \rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}}, \tag{10}$$

where

$$N_{K,\varrho_D}(m) := \sup_{\mathbf{x} \in D} N_{K,\varrho_D}(m, \mathbf{x}) = \sup_{\mathbf{x} \in D} \sum_{k=1}^{m-1} |\eta_k(\mathbf{x})|^2 .$$

$$\|P_{m-1}f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)} = \|\widetilde{S}_{\mathbf{X}}^m (f - P_{m-1}f)\|_{\ell_\infty(D)} = \left\|\sum_{k=1}^{m-1} \tilde{c}_k \eta_k(\mathbf{x})\right\|_{\ell_\infty(D)}$$

$$\leq \sqrt{N_{K,\varrho_D}(m) \cdot \sum_{k=1}^{m-1} |\tilde{c}_k|^2}$$

$$\|P_{m-1}f - \widetilde{S}^m_{\mathbf{X}} f\|_{\ell_\infty(D)} = \|\widetilde{S}^m_{\mathbf{X}} (f - P_{m-1}f)\|_{\ell_\infty(D)} = \left\| \sum_{k=1}^{m-1} \tilde{c}_k \eta_k(\mathbf{x}) \right\|_{\ell_\infty(D)}$$

$$\leq \sqrt{N_{K,\varrho_D}(m) \cdot \sum_{k=1}^{m-1} |\tilde{c}_k|^2}$$

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$(\tilde{c}_1, \dots, \tilde{c}_{m-1})^\top = (\widetilde{\mathbf{L}}^*_{k,m} \widetilde{\mathbf{L}}_{k,m})^{-1} \widetilde{\mathbf{L}}^*_{k,m} \cdot \mathbf{g}, \quad \boldsymbol{g} = (g_j)_{j=1}^n$ with

$$g_j := \begin{cases} 0, & \varrho_m(\mathbf{x}^j) = 0, \\ (f - P_{m-1}f)(\mathbf{x}^j)/\sqrt{\varrho_m(\mathbf{x}^j)}, & \varrho_m(\mathbf{x}^j) \neq 0\,. \end{cases} \tag{11}$$

**N. Nagel, M. Schäfer, T. Ullrich** A new upper bound for sampling numbers. *Found. Comp. Math.*, 2021.

$$N(m) \leq n/(10 r \log n), \ r > 1 \quad \Longrightarrow \quad \left\| (\mathbf{L}^*_{n,m} \mathbf{L}_{n,m})^{-1} \mathbf{L}^*_{n,m} \right\|_{2\to 2} \leq \sqrt{2/n}$$

$$\varrho_m(\mathbf{x}) = \frac{1}{2}\left( \frac{1}{m-1} \sum_{k=1}^{m-1} |\eta_k(\mathbf{x})|^2 + \frac{1}{\sum_{k=m}^{\infty} \lambda_k} \sum_{k=m}^{\infty} |e_k^*(\mathbf{x})|^2 \right)$$

$$\widetilde{N}(m) := \sup_{\mathbf{x}\in D} \sum_{k=1}^{m-1} \frac{|\eta_k(\mathbf{x})|^2}{\varrho_m(\mathbf{x})} \leq 2(m-1)$$

For systems $(\eta_k(\mathbf{x}))_{k=1}^{\infty}$, where for all $k \in \mathbb{N}$

$$\|\eta_k\|_{\ell_\infty(D)} \leq B, \quad k \in \mathbb{N},$$

we have

$$N_{K,\varrho_D}(m) \leq (m-1)B^2 \tag{12}$$

## Theorem (Moeller, Ullrich' 20)

Let $\mathbf{y}^i$, $i = 1, \ldots, n$, be i.i.d random sequences from $\ell_2$. Let further $n \geq 3$, $r > 1$, $M > 0$ such that $\|\mathbf{y}^i\|_2 \leq M$ for all $i = 1, \ldots, n$ almost surely and $\mathbb{E}\mathbf{y}^i \otimes \mathbf{y}^i = \mathbf{\Lambda}$ for $i = 1, \ldots, n$ with $\|\mathbf{\Lambda}\|_{2\to 2} \leq 1$. Then

$$\mathbb{P}\left(\left\|\frac{1}{n}\sum_{i=1}^{n}\mathbf{y}^i \otimes \mathbf{y}^i - \mathbf{\Lambda}\right\|_{2\to 2} \geq F\right) \leq 2^{3/4} n^{1-r},$$

where $F := \max\left\{\frac{8r\log n}{n} M^2\kappa^2, \|\mathbf{\Lambda}\|_{2\to 2}\right\}$ and $\kappa = \frac{1+\sqrt{5}}{2}$.

Focus here on **infinite** random matrices, complements earlier results by Kämmerer, Ullrich, Volkmer, Tropp, Rauhut, Pajor, Mendelson, Oliveira...

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$\mathbf{y}^i := \frac{1}{\sqrt{\varrho_m(\mathbf{x}^i)}}(e_m^*(\mathbf{x}^i), e_{m+1}^*(\mathbf{x}^i), \ldots)$, $i = 1, \ldots, n$

$$\|\mathbf{y}^i\|_2^2 \leq \sup_{\mathbf{x}\in D}\sum_{k=m}^{\infty}\frac{|e_k^*(\mathbf{x})|^2}{\varrho_m(\mathbf{x})} \leq 2\sum_{k=m}^{\infty}\lambda_k =: M$$

$\mathbf{\Lambda} := \mathrm{diag}(\sigma_m^2, \sigma_{m+1}^2, \ldots)$, $\|\mathbf{\Lambda}\|_{2\to 2} = \sigma_m^2$

## Theorem (P., Ullrich' 21)

- $H(K)$ RKHS on a compact domain $D \subset \mathbb{R}^d$
- $K\colon D \times D \to \mathbb{C}$ continuous and bounded kernel
- $\varrho_D$ finite Borel measure with full support on $D$
- $(\sigma_n)_{n=1}^{\infty}$, $\sigma_1 \geq \sigma_2 \geq \cdots$, singular values of $\mathrm{Id}\colon H(K) \to L_2(D, \varrho_D)$
- $m := \lfloor n/(c_1 r \log n) \rfloor$, $r > 1$

$$\sup_{\|f\|_{H(K)} \leq 1} \|f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)}^2 \leq c_3 \max \left\{ \frac{N_{K,\varrho_D}(m)}{m} \sum_{k \geq \lfloor m/2 \rfloor} \sigma_k^2, \sum_{k \geq \lfloor m/2 \rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k} \right\} \quad (13)$$

with probability larger than $1 - c_2 n^{1-r}$, where $\mathbf{X} = (\mathbf{x}^1, \ldots, \mathbf{x}^n)$ are drawn i.i.d. with respect to $\varrho_m(\cdot) d\varrho_D$, $N_{K,\varrho_D}(m) = \sup_{\mathbf{x} \in D} \sum_{k=1}^{m-1} \left| \sigma_k^{-1} e_k^*(\mathbf{x}) \right|^2$, $(e_n^*(\mathbf{x}))_{n=1}^{\infty} \subset H(K)$.

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- $m := \lfloor n/(c_1 r \log n) \rfloor$, $r > 1$

$$\sup_{\|f\|_{H(K)} \leq 1} \|f - \widetilde{S}^m_{\mathbf{X}} f\|^2_{\ell_\infty(D)} \leq c_3 \max\left\{ \frac{N_{K,\varrho_D}(m)}{m} \sum_{k \geq \lfloor m/2 \rfloor} \sigma_k^2, \sum_{k \geq \lfloor m/2 \rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k} \right\} \quad (13)$$

with probability larger than $1 - c_2 n^{1-r}$, where $\mathbf{X} = (\mathbf{x}^1, ..., \mathbf{x}^n)$ are drawn i.i.d. with respect to $\varrho_m(\cdot) d\varrho_D$, $N_{K,\varrho_D}(m) = \sup_{\mathbf{x} \in D} \sum_{k=1}^{m-1} \left|\sigma_k^{-1} e_k^*(\mathbf{x})\right|^2$, $(e_n^*(\mathbf{x}))_{n=1}^{\infty} \subset H(K)$.

$$c_2 = 3, \ c_3 = 8(4(1+\sqrt{5})/\sqrt{c_1} + 3)^2$$

Arbitrary ONS $(\eta_k(\mathbf{x}))_{k=1}^{\infty}$
$c_1 = 20, \ c_3 = 278$

$\|\eta_k\|_{\ell_\infty(D)} \leq 1, \ k \in \mathbb{N}$
$c_1 = 10, \ c_3 = 403$

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- $K\colon D \times D \to \mathbb{C}$ continuous and bounded kernel
- $\varrho_D$ finite Borel measure with full support on $D$
- $(\sigma_n)_{n=1}^{\infty}$, $\sigma_1 \geq \sigma_2 \geq \cdots$, singular values of $\mathrm{Id}\colon H(K) \to L_2(D, \varrho_D)$
- $m := \lfloor n/(c_1 r \log n) \rfloor$, $r > 1$

$$\sup_{\|f\|_{H(K)} \leq 1} \|f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)}^2 \leq c_3 \max\left\{ \frac{N_{K,\varrho_D}(m)}{m} \sum_{k \geq \lfloor m/2 \rfloor} \sigma_k^2, \sum_{k \geq \lfloor m/2 \rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k} \right\} \qquad (14)$$

with probability larger than $1 - c_2 n^{1-r}$, where $\mathbf{X} = (\mathbf{x}^1, \ldots, \mathbf{x}^n)$ are drawn i.i.d. with respect to $\varrho_m(\cdot) d\varrho_D$, $N_{K,\varrho_D}(m) = \sup_{\mathbf{x} \in D} \sum_{k=1}^{m-1} \left|\sigma_k^{-1} e_k^*(\mathbf{x})\right|^2$, $(e_n^*(\mathbf{x}))_{n=1}^{\infty} \subset H(K)$.

If $N_{K,\varrho_D}(k) = \mathcal{O}(k)$, it holds

$$\sup_{\|f\|_{H(K)} \leq 1} \|f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)}^2 \leq C_{\varrho_D,K} \sum_{k \geq \lfloor m/2 \rfloor} \sigma_k^2 \leq C_{\varrho_D,K}\, a_{\lfloor m/2 \rfloor}(\mathrm{Id}_{K,\infty})^2 \qquad (15)$$

**F. Cobos, T. Kühn, W. Sickel** Optimal approximation of multivariate periodic Sobolev functions in the sup-norm. *J. Funct. Anal.*, 270(11):4196–4212, 2016.

$$m := \lfloor n/(c_1 r \log n) \rfloor, \quad r > 1$$

$$\varrho_m(\boldsymbol{x}) = \frac{1}{2}\left(\frac{1}{m-1}\sum_{k=1}^{m-1}|\eta_k(\boldsymbol{x})|^2 + \frac{1}{\sum_{k=m}^{\infty}\lambda_k}\sum_{k=m}^{\infty}|e_k^*(\boldsymbol{x})|^2\right)$$

$$\sup_{\|f\|_{H(K)}\le 1}\|f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)}^2 \le c_3 \max\left\{\frac{N_{K,\varrho_D}(m)}{m}\sum_{k\ge\lfloor m/2\rfloor}\sigma_k^2, \sum_{k\ge\lfloor m/2\rfloor}\frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}\right\}$$

$$\varrho_m'(\boldsymbol{x}) = \frac{1}{2(m-1)}\sum_{k=1}^{m-1}|\eta_k(\boldsymbol{x})|^2 + \frac{1}{2}$$

$$\sup_{\|f\|_{H(K)}\le 1}\|f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)}^2 \le c_3 \max\left\{\frac{N_{K,\varrho_D}(m)}{m}\sum_{k\ge\lfloor cm\rfloor}\frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}, \sum_{k\ge\lfloor m/2\rfloor}\frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}\right\}$$

$$m := \lfloor n/(c_1 r \log n) \rfloor, \quad r > 1$$

$$\varrho_m(\boldsymbol{x}) = \frac{1}{2}\left(\frac{1}{m-1}\sum_{k=1}^{m-1}|\eta_k(\boldsymbol{x})|^2 + \frac{1}{\sum_{k=m}^{\infty}\lambda_k}\sum_{k=m}^{\infty}|e_k^*(\boldsymbol{x})|^2\right)$$

$$\sup_{\|f\|_{H(K)}\le 1}\|f - \widetilde{S}_{\mathbf{X}}^m f\|^2_{\ell_\infty(D)} \le c_3 \max\left\{\frac{N_{K,\varrho_D}(m)}{m}\sum_{k\ge\lfloor m/2\rfloor}\sigma_k^2, \sum_{k\ge\lfloor m/2\rfloor}\frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}\right\}$$

$$\varrho'_m(\boldsymbol{x}) = \frac{1}{2(m-1)}\sum_{k=1}^{m-1}|\eta_k(\boldsymbol{x})|^2 + \frac{1}{2}$$

$$\sup_{\|f\|_{H(K)}\le 1}\|f - \widetilde{S}_{\mathbf{X}}^m f\|^2_{\ell_\infty(D)} \le c_3 \max\left\{\frac{N_{K,\varrho_D}(m)}{m}\sum_{k\ge\lfloor cm\rfloor}\frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}, \sum_{k\ge\lfloor m/2\rfloor}\frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}\right\}$$

$m^*$ is the largest number such that $N_{K,\varrho_D}(m) \le n/(10 r \log n)$

$$\sup_{\|f\|_{H(K)}\le 1}\|f - S_{\mathbf{X}}^{m^*} f\|^2_{\ell_\infty(D)} \le C\sum_{k\ge\lfloor m^*/2\rfloor}\frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k} \tag{16}$$

$$\mathbf{L}_{n,m} := \mathbf{L}_{n,m}(\mathbf{X}) = \begin{pmatrix} \eta_1(\mathbf{x}^1) & \eta_2(\mathbf{x}^1) & \cdots & \eta_{m-1}(\mathbf{x}^1) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_1(\mathbf{x}^n) & \eta_2(\mathbf{x}^n) & \cdots & \eta_{m-1}(\mathbf{x}^n) \end{pmatrix}$$

Above approach requires $n = \mathcal{O}(m \log m)$ samples.

We "shrink" the matrix $\mathbf{L}_{n,m}$ to $\mathcal{O}(m)$ lines applying a modification of the Weaver sub-sampling strategy.

$$\widetilde{S}^m_{\mathbf{X}}, \ \mathbf{X} = \{\mathbf{x}^1, \dots, \mathbf{x}^n\} \quad \Longrightarrow \quad \widetilde{S}^m_J, \ \#J = \mathcal{O}(m), \ (\mathbf{x}^i)_{i \in J} \subset \mathbf{X}$$

**N. Nagel, M. Schäfer, T. Ullrich** A new upper bound for sampling numbers. *Found. Comp. Math.*, 2021.

**S. Nitzan, A. Olevskii, A. Ulanovskii** Exponential frames on unbounded sets. *Proc. Amer. Math. Soc.*, 144(1):109–118, 2016.

**I. Limonova, V. N. Temlyakov** On sampling discretization in $L_2$. *arXiv: math/2009.10789v1*, 2020.

# Theorem (Nitzan, Olevskii, Ulanovskii' 16, Limonova, Temlyakov' 20, Nagel, Schäfer, Ullrich' 20)

Let $k_1, k_2, k_3 > 0$ and $\mathbf{u}_1, ..., \mathbf{u}_n \in \mathbb{C}^m$ with $\|\mathbf{u}_i\|_2^2 \le k_1 \frac{m}{n}$ for all $i = 1, ..., n$ and

$$k_2 \|\mathrm{w}\|_2^2 \le \sum_{i=1}^{n} |\langle \mathbf{w}, \mathbf{u}_i \rangle|^2 \le k_3 \|\mathbf{w}\|_2^2 , \quad \mathbf{w} \in \mathbb{C}^m$$

$\Longrightarrow \quad \exists J \subseteq \{1, \ldots, n\},\ \#J \le C_1 m$:

$$C_2 \cdot \frac{m}{n} \|\mathbf{w}\|_2^2 \le \sum_{i \in J} |\langle \mathbf{w}, \mathbf{u}_i \rangle|^2 \le C_3 \cdot \frac{m}{n} \|\mathbf{w}\|_2^2, \quad \mathbf{w} \in \mathbb{C}^m$$

More precisely, we can choose

$$C_1 = 1642 \frac{k_1}{k_2} , \quad C_2 = (2 + \sqrt{2})^2 k_1 , \quad C_3 = 1642 \frac{k_1 k_3}{k_2}$$

in case $\frac{n}{m} \ge 47 \frac{k_1}{k_2}$. In the regime $1 \le \frac{n}{m} < 47 \frac{k_1}{k_2}$ one may put $C_1 = 47 \frac{k_1}{k_2}$, $C_2 = k_2$, $C_3 = 47 \frac{k_1 k_3}{k_2}$.

# Theorem (P., Ullrich' 21)

For $\mathrm{Id}\colon H(K) \to \ell_\infty(D)$, $\exists b, c_4, c_5, c_6 > 0$:

$$g_{\lfloor bm \log m \rfloor}(\mathrm{Id})^2 \le c_3 \max\left\{ \frac{N_{K,\varrho_D}(m)}{m} \sum_{k \ge \lfloor m/2 \rfloor} \sigma_k^2 \,, \sum_{k \ge \lfloor m/2 \rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k} \right\}$$

$$g_m(\mathrm{Id})^2 \le c_4 \max\left\{ \frac{N_{K,\varrho_D}(m) \log m}{m} \sum_{k \ge \lfloor c_5 m \rfloor} \sigma_k^2 \,, \sum_{k \ge \lfloor c_5 m \rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k} \right\}$$

The measure $\varrho_D$ is at our disposal.

# Theorem (P., Ullrich' 21)

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The measure $\varrho_D$ is at our disposal.

If $N_{K,\varrho_D}(k) = \mathcal{O}(k)$ we obtain

$$g_m(\mathrm{Id}) \leq C_{\varrho_D,K} \min\{a_{\lfloor m/(c_6 \log m) \rfloor}(\mathrm{Id})\,,\ \sqrt{\log m} \cdot a_{\lfloor c_5 m \rfloor}(\mathrm{Id})\} \tag{17}$$

# Sampling and Kolmogorov numbers

**D. Krieg and M. Ullrich**

- Function values are enough for $L_2$-approximation
- Function values are enough for $L_2$-approximation: Part II

**L. Kämmerer, T. Ullrich, T. Volkmer** Worst case recovery guarantees for least squares approximation using random samples

**M. Ullrich** On the worst-case error of least squares algorithms for $L_2$-approximation with high probability

**M. Moeller, T. Ullrich** $L_2$-norm sampling discretization and recovery of functions from RKHS with finite trace.

**N. Nagel, M. Schäfer, T. Ullrich** A new upper bound for sampling numbers

$$g_m(\mathrm{Id}\colon F \to L_2(D,\varrho))^2 \leq C\,\frac{\log m}{m} \sum_{k \geq \lfloor cm \rfloor} d_k(\mathrm{Id}\colon F \to L_2(D,\varrho))^2$$

**V. N. Temlyakov** On optimal recovery in $L_2$

$$g_{\lfloor bm \rfloor}(\mathrm{Id}\colon F \to L_2(D,\varrho)) \leq B\, d_m(\mathrm{Id}\colon F \to L_\infty(D))$$

$\mathrm{Id}\colon F \to \ell_\infty(D)$

$$d_m(\mathrm{Id}) := \inf_{V_m} \sup_{\|f\|_F \le 1} \inf_{g \in V_m} \|f - g\|_{\ell_\infty(D)}$$

- $V_m^*$ is the optimal subspace for $d_m(\mathrm{Id})$
- $\varrho$ is a finite measure on $D$
- $(\phi_n)_{n=1}^\infty$ is ONS in $V_m^*$ w.r.t. $\varrho$

$$g_{\lfloor bm \log m \rfloor}(\mathrm{Id}) \le \left(2 + \sqrt{\varrho(D)/(m-1)}\right)\sqrt{N_{\varrho, V_m^*}}\, d_m(\mathrm{Id})$$

$$g_{\lfloor bm \rfloor}(\mathrm{Id}) \le C \sqrt{N_{\varrho, V_m^*}}\, d_m(\mathrm{Id}), \quad m \ge \varrho(D)$$

where

$$N_{\varrho, V_m^*} := \sup_{f \in V_m^*} \frac{\|f\|_{\ell_\infty(D)}^2}{\|f\|_{L_2(D,\varrho)}^2}$$

**V. N. Temlyakov** On optimal recovery in $L_2$. *J. Complexity*, to appear. arXiv: math/2010.03103.

$$g_{\lfloor bm \rfloor}(\mathrm{Id}\colon F \to L_2(D,\varrho)) \le B\, d_m(\mathrm{Id}\colon F \to L_\infty(D))$$

**V. N. Temlyakov, T. Ullrich**

- Approximation of functions with small mixed smoothness in the uniform norm. *arXiv: math/2012.11983*, 2020.
- Bounds on Kolmogorov widths and sampling recovery for classes with small mixed smoothness. *arXiv: math/2012.09925*, 2020.

# The power of standard information

$\mathrm{Id}\colon H(K) \to F$

$$q_F^{\mathrm{lin}} := \sup\Big\{q \geq 0\colon \quad \lim_{n\to\infty} n^q a_n(\mathrm{Id}) = 0\Big\}$$

$$q_F^{\mathrm{std}} := \sup\Big\{q \geq 0\colon \quad \lim_{n\to\infty} n^q g_n(\mathrm{Id}) = 0\Big\}$$

$$F = L_2(D, \varrho_D) \implies q_{2,\varrho_D}^{\mathrm{lin}} := q_{L_2(D,\varrho_D)}^{\mathrm{lin}}, \; q_{2,\varrho_D}^{\mathrm{std}} := q_{L_2(D,\varrho_D)}^{\mathrm{std}}$$

$$F = \ell_\infty(D) \implies q_\infty^{\mathrm{lin}} := q_{\ell_\infty(D)}^{\mathrm{lin}}, \qquad q_\infty^{\mathrm{std}} := q_{\ell_\infty(D)}^{\mathrm{std}}$$

For $a_n(\mathrm{Id})$ one can take $F = L_\infty(D)$

# Assumptions

(i) $N_{K,\varrho_D}(k) = \mathcal{O}(k^u)$;

(ii) $\exists p > 1/2\,,\ C_2 > 0: \qquad \sigma_j \le C_2 j^{-p},\ j = 1, 2, \ldots.$

$$u := \inf\left\{u\colon\ N_{K,\varrho_D}(k) = \mathcal{O}(k^u)\right\},$$
$$p := \sup\{p\colon\ \sigma_j \le C_2 j^{-p},\ j = 1, 2, \ldots\}.$$

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**F. Kuo, G. W. Wasilkowski, H. Woźniakowski**

- Multivariate $L_\infty$ approximation in the worst case setting over reproducing kernel Hilbert spaces. *J. Approx. Theory*, 152(2):135–160, 2008.
- On the power of standard information for multivariate approximation in the worst case setting. *J. Approx. Theory*, 158(1):97–125, 2009.


(i') $\exists C_1 > 0 \qquad \|\eta_j\|_{\ell_\infty(D)} \leq C_1,\ j = 1, 2, \ldots$

$$N_{K,\varrho_D}(k) = \sup_{\mathbf{x}\in D} \sum_{i=1}^{k-1} |\eta_i(\mathbf{x})|^2 = \mathcal{O}(k),\ \ \eta_i(\mathbf{x}) = \sigma_i^{-1} e_i^*(\mathbf{x})$$

# The power of standard information

(i) $N_{K,\varrho_D}(k) = \mathcal{O}(k^u)$;

(i') $\exists C_1 > 0$: $\|\eta_j\|_{\ell_\infty(D)} \le C_1$, $j = 1, 2, \ldots$ $(u = 1)$

(ii) $\exists p > 1/2$, $C_2 > 0$: $\sigma_j \le C_2 j^{-p}$, $j = 1, 2, \ldots$

$$\begin{aligned} \text{(ii)} \quad &\Longrightarrow \quad q^{\mathrm{lin}}_{2,\varrho_D} = p \\ \text{(i}'\text{)} / \text{(i)} \;\; \& \;\; \text{(ii)} \quad &\Longrightarrow \quad q^{\mathrm{lin}}_{\infty} = q^{\mathrm{lin}}_{2,\varrho_D} - 1/2 = p - 1/2 \\ \text{(i}'\text{)} \;\; \& \;\; \text{(ii)} \quad &\Longrightarrow \quad q^{\mathrm{std}}_{\infty} \in \left[ \frac{2p}{2p+1} \left( p - \frac{1}{2} \right), p - \frac{1}{2} \right] \end{aligned}$$

## Corollary (P., Ullrich' 21)

(i) & (ii), $2p > u$

$$q^{\mathrm{std}}_{\infty} \ge p - u/2$$

In case $u = 1$ (or if (i') holds) we have

$$q^{\mathrm{std}}_{\infty} = q^{\mathrm{lin}}_{\infty} = p - 1/2$$

# Examples. Sobolev type spaces.

$H^w$, $w(\boldsymbol{k}) > 0$, $\boldsymbol{k} \in \mathbb{Z}^d$

$$\|f\|^2_{H^w(\mathbb{T}^d)} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (w(\boldsymbol{k}))^2 |c_{\boldsymbol{k}}(f)|^2 < \infty \tag{18}$$

$c_{\boldsymbol{k}}(f) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(\mathrm{x}) e^{-i\boldsymbol{k}\cdot\mathrm{x}} \mathrm{dx}$, $\quad \boldsymbol{k} \in \mathbb{Z}^d$, $\mathbb{T}^d = [0, 2\pi]^d$

---

$$H^w(\mathbb{T}^d) \hookrightarrow L_\infty(\mathbb{T}^d) \quad \Longleftrightarrow \quad \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (w(\boldsymbol{k}))^{-2} < \infty$$

**F. Cobos, T. Kühn, W. Sickel** Optimal approximation of multivariate periodic Sobolev functions in the sup-norm. *J. Funct. Anal.*, 270(11):4196–4212, 2016.

$$K_w(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{(w(\boldsymbol{k}))^2} \tag{19}$$

$H^w(\mathbb{T}^d) \hookrightarrow \ell_\infty(\mathbb{T}^d)$

$$K_w(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{(w(\boldsymbol{k}))^2}$$

$(\lambda_n)_{n\in\mathbb{N}} = ((w(\boldsymbol{k}))^{-2})_{\boldsymbol{k}\in\mathbb{Z}^d}, \;\; (\eta_n(\cdot))_{n\in\mathbb{N}} = (\exp(i\boldsymbol{k}\cdot))_{\boldsymbol{k}\in\mathbb{Z}^d}$

---

$$I(R) := \left\{\boldsymbol{k}\in\mathbb{Z}^d : w(\boldsymbol{k}) \le R\right\}, \quad m(R) = \# I(R)$$

## Theorem (P., Ullrich' 21)

- $H^w$, $\sum_{\boldsymbol{k}\in\mathbb{Z}^d}(w(\boldsymbol{k}))^{-2} < \infty$
- $R \ge 1$, $r > 1$
- *n be smallest such that* $m(R) \le \lfloor n/(c_1 r \log n)\rfloor$, $c_1 > 0$

$$\sup_{\|f\|_{H(K_w)}\le 1} \|f - S_{\mathbf{X}}^{m(R)} f\|^2_{L_\infty(\mathbb{T}^d)} \le C \sum_{\boldsymbol{k}:\, w(\boldsymbol{k})>R} (w(\boldsymbol{k}))^{-2} \qquad (20)$$

# Sampling recovery. Sobolev type spaces.

$$g_n(\mathrm{I}_w : H^w(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)) \le C \min\left\{ a_{\lfloor n/b \log n \rfloor}(\mathrm{I}_w)\,,\, \sqrt{\log n} \cdot a_{\lfloor cn \rfloor}(\mathrm{I}_w) \right\}$$

Note that

$$a_n(\mathrm{I}_w)^2 = \sum_{k \ge n+1} \tau_k^2\,,$$

where $(\tau_n)_{n\in\mathbb{N}} = (1/w(\boldsymbol{k}))_{\boldsymbol{k}\in\mathbb{Z}^d}, \quad \tau_1 \ge \tau_2 \ge \dots$

---

$$g_m(\mathrm{I}_w) \le C \log m \cdot a_{\lfloor m/\log m \rfloor}(\mathrm{I}_w) \tag{21}$$


**L. Kämmerer** Multiple lattice rules for multivariate $L_\infty$ approximation in the worst-case setting. *arXiv: math/1909.02290v1*, 2019.

**L. Kämmerer, T. Volkmer** Approximation of multivariate periodic functions based on sampling along multiple rank-1 lattices. *J. Approx. Theory*, 246:1–27, 2019.

$$H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d) := \left\{ f \in L_2(\mathbb{T}^d)\colon \ \|f\|^2_{H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d)} = \sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \prod_{j=1}^{d} (1+|k_j|)^{2s} < \infty \right\}$$

$$H^{s,+}_{\mathrm{mix}}(\mathbb{T}^d) := \left\{ f \in L_2(\mathbb{T}^d)\colon \ \|f\|^2_{H^{s,+}_{\mathrm{mix}}(\mathbb{T}^d)} = \sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \prod_{j=1}^{d} \left(1+|k_j|^2\right)^{s} < \infty \right\}$$

**T. Kühn, W. Sickel, T. Ullrich** Approximation of mixed order Sobolev functions on the $d$-torus: asymptotics, preasymptotics and $d$-dependence. *Constr. Approx.*, 42:353–398, 2015.

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$$\mathrm{Id}\colon H^s_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d), \quad \sigma_n \lesssim_{s,d} n^{-s} (\log n)^{s(d-1)}$$

## Theorem (P., Ullrich' 21)

*Let* $s > 1/2$, $r > 1$, $m = \lfloor n/(c_1 r \log n) \rfloor$, $c_1 > 0$. *Then*

$$\sup_{\|f\|_{H^s_{\mathrm{mix}}(\mathbb{T}^d)} \le 1} \|f - S^m_{\mathbf{X}} f\|_{L_\infty(\mathbb{T}^d)} \lesssim_{s,d,r} n^{-s+1/2} (\log n)^{sd-1/2} \qquad (22)$$

*is true as* $n \to \infty$ *with probability larger than* $1 - 3n^{1-r}$.

$$g_{\lfloor bm\log m\rfloor}(\mathrm{Id}\colon H^s_{\mathrm{mix}}(\mathbb{T}^d)\to L_\infty(\mathbb{T}^d)) \lesssim_{s,d} m^{-s+1/2}(\log m)^{s(d-1)} \tag{23}$$

$$g_{\lfloor cn\rfloor}(\mathrm{Id}\colon H^s_{\mathrm{mix}}(\mathbb{T}^d))\to L_\infty(\mathbb{T}^d)) \lesssim_{s,d} n^{-s+1/2}(\log n)^{s(d-1)+1/2} \tag{24}$$

$$s(d-1)+1/2 < s(d-1) + s - 1/2 \quad \text{if} \quad s>1$$

---

For sparse grids

$$g_n(\mathrm{Id}\colon H^s_{\mathrm{mix}}(\mathbb{T}^d)\to L_\infty(\mathbb{T}^d)) \asymp_{s,d} n^{-s+1/2}(\log n)^{(d-1)s}, \quad s>1/2$$

**V. N. Temlyakov** On approximate recovery of functions with bounded mixed derivative. *J. Complexity*, 9(1):41–59, 1993.

$$\sigma_n^{\#} \le \left(\frac{16}{3n}\right)^{\frac{s}{1+\log_2 d}}, \quad n \ge 6$$

**T. Kühn** New preasymptotic estimates for the approximation of periodic Sobolev functions. In *2018 MATRIX annals*, volume 3 of *MATRIX Book Ser.*, pages 97–112. Springer, Cham., 2020.

## Theorem (P., Ullrich' 21)

- $s > (1+\log_2 d)/2$, $\beta := 2s/(1+\log_2 d) > 1$, $r > 1$
- $n \in \mathbb{N}$, $n \ge 3$, $m \in \mathbb{N}$ $m = \lfloor n/(10r\log n) \rfloor$

$$\sup_{\|f\|_{H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d)} \le 1} \|f - S^m_{\mathbf{X}} f\|^2_{L_\infty(\mathbb{T}^d)} \le 1612 \left(\frac{16}{3}\right)^{\beta} \frac{\beta}{\beta-1} \left(\frac{m}{2} - 1\right)^{-\beta+1} \qquad (25)$$

$$\sigma_n^{\#} \le \left(\frac{16}{3n}\right)^{\frac{s}{1+\log_2 d}}, \quad n \ge 6$$

**T. Kühn** New preasymptotic estimates for the approximation of periodic Sobolev functions. In *2018 MATRIX annals*, volume 3 of *MATRIX Book Ser.*, pages 97–112. Springer, Cham., 2020.

## Theorem (P., Ullrich' 21)

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$$\sup_{\|f\|_{H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d)} \le 1} \|f - S^m_{\mathbf{X}} f\|^2_{L_\infty(\mathbb{T}^d)} \le 1612 \left(\frac{16}{3}\right)^{\beta} \frac{\beta}{\beta - 1} \left(\frac{m}{2} - 1\right)^{-\beta+1} \quad (25)$$

$$\sigma_n^{+} \le \left(\frac{C(d)}{n}\right)^{\frac{s}{2(1+\log_2(d-1))}}, \quad (26)$$

$s > 0$, $d \ge 3$, $n \ge 2$, $C(d) = \left(1 + \frac{1}{d-1}\left(1 + \frac{2}{\log_2(d-1)}\right)\right)^{d-1}$

**T. Kühn, W. Sickel, T. Ullrich** How anisotropic mixed smoothness affects the decay of singular numbers of Sobolev embeddings. *J. Complexity*, to appear, arXiv: math/2001.09022v3.

# General results

For $\mathrm{Id}\colon H(K) \to \ell_\infty(D)\,,\ \exists b, c_4, c_5, c_6 > 0:$

$$g_{\lfloor bm \log m \rfloor}(\mathrm{Id})^2 \le c_3 \max\left\{ \frac{N_{K,\varrho_D}(m)}{m} \sum_{k \ge \lfloor m/2 \rfloor} \sigma_k^2\,,\ \sum_{k \ge \lfloor m/2 \rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k} \right\}$$

$$g_m(\mathrm{Id})^2 \le c_4 \max\left\{ \frac{N_{K,\varrho_D}(m) \log m}{m} \sum_{k \ge \lfloor c_5 m \rfloor} \sigma_k^2\,,\ \sum_{k \ge \lfloor c_5 m \rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k} \right\}$$

# A univariate example

$D = [-1, 1]$, the uniform measure $\mathrm{d}x$ on $D$

$$Af(x) = -((1 - x^2)f')'$$

$$H(K_s) := \{f \in L_2(D)\colon\ A^{s/2}f \in L_2(D)\}, \quad s > 1$$

$$K_s(x, y) = \sum_{k \in \mathbb{N}} (1 + (k(k+1))^s)^{-1} \mathcal{P}_k(x)\mathcal{P}_k(y)$$

$\mathcal{P}_k\colon D \to \mathbb{R}$, $k \in \mathbb{N}$, are $L_2(D)$-normalized Legendre polynomials $\mathcal{P}_k(x)$

- $(\eta_k)_{k=1}^{\infty} = (\mathcal{P}_k)_{k=1}^{\infty}$
- $(e_k^*)_{k=1}^{\infty} = ((1 + (k(k+1))^s)^{-1/2}\mathcal{P}_k)_{k=1}^{\infty}$
- $\sigma_k = ((1 + (k(k+1))^s)^{-1/2}$

$$N(m) = \mathcal{O}(m^2)$$

**P. G. Nevai** Orthogonal polynomials. *Mem. Amer. Math. Soc.*, 18(213), 1979.

$$g_n(\mathrm{Id}) \lesssim_s n^{-s+1} (\log n)^{\min\{s-1, 1/2\}} \tag{27}$$

# A univariate example

$$\sup_{\|f\|_{H(K_s)}\leq 1} \|f - \widetilde{S}_{\mathrm{X}}^m f\|_{L_\infty(D)} \lesssim_s m^{-s+1} \lesssim n^{-s+1} (\log n)^{s-1}$$

$L_2(D)$, Gauss points

**C. Bernardi, Y. Maday** Polynomial interpolation results in Sobolev spaces. *J. Comput. Appl. Math.*, 43(1):53–80, 1992.

$L_2(D)$, worst case error estimates with high probability

**L. Kämmerer, T. Ullrich, T. Volkmer** Worst-case recovery guarantees for least squares approximation using random samples. *arXiv: math/1911.10111*, 2019.

$$\sup_{\|f\|_{H(K_s)}\le 1} \|f - \widetilde{S}^m_{\mathbf{X}} f\|_{L_\infty(D)} \lesssim_s m^{-s+1} \lesssim n^{-s+1} (\log n)^{s-1}$$

$$\varrho_m(\mathbf{x}) = \frac{1}{2}\Big(\frac{1}{m-1}\sum_{k=1}^{m-1} |\eta_k(\mathbf{x})|^2 + \frac{1}{\sum_{k=m}^{\infty}\lambda_k}\sum_{k=m}^{\infty} |e_k^*(\mathbf{x})|^2\Big) \qquad (28)$$

$$\sup_{\|f\|_{H(K_s)}\le 1} \|f - \widetilde{S}^m_{\mathbf{X}} f\|_{L_\infty(D)} \lesssim_s n^{-s+3/2} (\log n)^{s-3/2}$$

$$\varrho'_m(\mathbf{x}) = \frac{1}{2(m-1)}\sum_{k=1}^{m-1} |\eta_k(\mathbf{x})|^2 + \frac{1}{2} \qquad (29)$$

$$\sup_{\|f\|_{H(K_s)}\le 1} \|f - S^m_{\mathbf{X}} f\|_{L_\infty(D)} \lesssim_s n^{-(s-1)/2} (\log n)^{(s-1)/2}$$

# Chebyshev kernels

$L_2^{\vartheta}(\mathbb{I}^d), \ \mathbb{I}^d = [-1,1]^d$

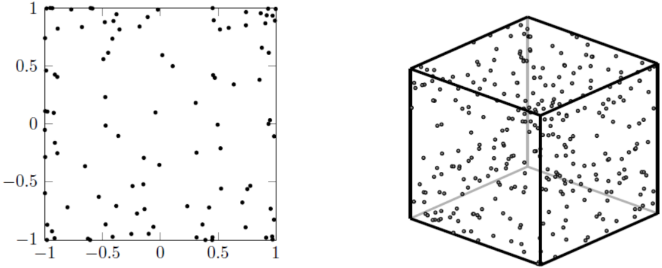
$$\|f\|_{L_2^{\vartheta}(\mathbb{I}^d)} := \left( \int_{\mathbb{I}^d} |f(\boldsymbol{x})|^2 \vartheta(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right)^{1/2} = \left( \int_{\mathbb{I}^d} |f(\boldsymbol{x})|^2 \prod_{j=1}^{d} \frac{1}{\pi \sqrt{1-x_j^2}} \mathrm{d}\boldsymbol{x} \right)^{1/2}$$

$L_2^{\theta}(\mathbb{I}^1)$: $\quad \{T_k(x)\}_{k\in\mathbb{N}_0} = \{1\} \bigcup \left\{ \sqrt{2} \cos(k \arccos x) \right\}_{k\in\mathbb{N}}$

$L_2^{\vartheta}(\mathbb{I}^d)$: $\quad \{T_{\boldsymbol{k}}(\boldsymbol{x})\}_{\boldsymbol{k}\in\mathbb{N}_0^d}$, where

$$T_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^{d} T_{k_j}(x_j) = \prod_{j=1}^{d} \left( \sqrt{2}^{\min(1,k_j)} \cos(k_j \arccos x_j) \right), \quad \boldsymbol{k} \in \mathbb{N}_0^d$$

# Realizations of random nodes with respect to the Chebyshev measure




(a) $d = 2$, $n = 100$ (b) $d = 3$, $n = 316$

Taken from:
**L. Kämmerer, T. Ullrich, T. Volkmer** Worst case recovery guarantees for least squares approximation using random samples, arXiv: 1911.10111, 2019.

$f \in L_2^{\vartheta}(\mathbb{I}^d) \quad f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \hat{f}_{\boldsymbol{k}} T_{\boldsymbol{k}}(\boldsymbol{x})$, where $\hat{f}_{\boldsymbol{k}} = (f, T_{\boldsymbol{k}})_{L_2^{\vartheta}(\mathbb{I}^d)}$, $\boldsymbol{k} \in \mathbb{N}_0^d$.

$H(K_w)$:

$$K_w(\boldsymbol{x}, \boldsymbol{y}) := \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \frac{T_{\boldsymbol{k}}(\boldsymbol{x}) T_{\boldsymbol{k}}(\boldsymbol{y})}{(w(\boldsymbol{k}))^2}, \tag{30}$$

where $w(\boldsymbol{k})$ is such, that $w(\boldsymbol{k}) \le w(\boldsymbol{k}')$ if $\boldsymbol{k} \le \boldsymbol{k}'$.

$$(\sigma_n)_{n=1}^{\infty} = \left(\frac{1}{w(\boldsymbol{k})}\right)_{\boldsymbol{k}\in\mathbb{N}_0^d}$$

$$H(\ell) := \left\{\boldsymbol{k} \in \mathbb{N}_0^d : \ \frac{1}{w(\boldsymbol{k})} = \sigma_i, \ i = 1, \dots, \ell\right\}$$

$(\sigma_n^\pi)_{n=1}^{\infty} = \left(\frac{1}{w^\pi(\boldsymbol{k})}\right)_{\boldsymbol{k}\in\mathbb{Z}^d}$, where

$$w^\pi(\boldsymbol{k}) := w(|\boldsymbol{k}|),\ |\boldsymbol{k}| = (|k_1|, \dots, |k_d|),\ \boldsymbol{k} \in \mathbb{Z}^d.$$

$M(A)$ denotes a set, that is "mirrored" to the downward closed set $A$ of indexes $\boldsymbol{k} \in \mathbb{N}_0^d$

$$\sum_{\boldsymbol{k}\in A} 2^{|\boldsymbol{k}|_0} \le (\#A)^{\frac{\ln 3}{\ln 2}}, \qquad \#M(A) \le \min\left(2^d \#A, (\#A)^{\frac{\ln 3}{\ln 2}}\right) \tag{31}$$

**A. Chkifa, A. Cohen, G. Migliorati, F. Nobilem, R. Tempone** Discrete least squares polynomial approximation with random evaluations — application to parametric and stochastic elliptic PDEs, ESAIM Math. Model. Numer. Anal. **49** (2015), 815-837.

**F. Y. Kuo, G. Migliorati, F. Nobile, D. Nuyens** Function integration, reconstruction and approximation using rank-1 lattices, ArXiv: 1908.01178v1, 2019.

## Theorem.

- $H(K_w)$ on $\mathbb{I}^d = [-1,1]^d$ with $\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \frac{1}{(w(\boldsymbol{k}))^2} < \infty$
- $r \geq 2,\ m = \left\lfloor \frac{n}{10 r \ln n} \right\rfloor,\ \widetilde{m} = \left\lfloor \max\left(2^{-d} m, m^{\frac{\ln 2}{\ln 3}}\right) \right\rfloor$
- $\vartheta(\mathrm{x}) = \prod_{j=1}^d \frac{1}{\pi\sqrt{1-\mathrm{x}_j^2}}$

$$\mathbb{P}\left( \sup_{\|f\|_{H(K_w)}\leq 1} \|f - S_{\mathbf{X}}^{m'} f\|^2_{L_2^{\vartheta}(\mathbb{I}^d)} \leq C_1 \max\left\{ (\sigma^{\pi}_{\tilde{m}})^2, \frac{r\ln n}{n} \sum_{j=\tilde{m}}^{\infty} \left(\sigma_j^{\pi}\right)^2 \right\} \right) \geq 1 - 3n^{1-r},$$

$$\mathbb{P}\left( \sup_{\|f\|_{H(K_w)}\leq 1} \|f - S_{\mathbf{X}}^{m'} f\|^2_{\ell_{\infty}(\mathbb{I}^d)} \leq C_2 \max\left\{ m\,(\sigma^{\pi}_{\tilde{m}})^2, \sum_{j=\tilde{m}}^{\infty} \left(\sigma_j^{\pi}\right)^2 \right\} \right) \geq 1 - 3n^{1-r},$$

where $m'$ is a maximal natural number, that satisfies the condition

$$N(m') = \sum_{\boldsymbol{k}\in H(m'-1)} 2^{|\boldsymbol{k}|_0} \leq m.$$

Let $w_s(\boldsymbol{k}) := \prod_{j=1}^{d}(1+k_j)^s$, $\boldsymbol{k} \in \mathbb{N}_0^d$, $s > 1/2$

$$\tilde{H}^s_{\text{mix}}(\mathbb{I}^d) := \left\{ f \in L_2^{\vartheta}(\mathbb{I}^d) : \quad \|f\|_{\tilde{H}^s_{\text{mix}}}(\mathbb{I}^d) = \left( \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} |\hat{f}_{\boldsymbol{k}}|^2 \prod_{j=1}^{d}(1+k_j)^{2s} \right)^{1/2} < \infty \right\} \quad (32)$$

$H^{s,\#}_{\text{mix}}(\mathbb{T}^d)$, $\mathbb{T}^d = [0,2\pi]^d$, is the periodic Sobolev space,

$$H^{s,\#}_{\text{mix}}(\mathbb{T}^d) = \left\{ f \in L_2(\mathbb{T}^d) : \quad \|f\|_{H^{s,\#}_{\text{mix}}(\mathbb{T}^d)} = \left( \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |c_{\boldsymbol{k}}(f)|^2 \prod_{j=1}^{d}(1+|k_j|)^{2s} \right)^{1/2} < \infty \right\}, \quad (33)$$

$w_s^{\pi}(\boldsymbol{k}) := \prod_{j=1}^{d}(1+|k_j|)^s$, $\boldsymbol{k} \in \mathbb{Z}^d$, $s > 1/2$.

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Thank you
for your attention!

$$\sup_{\|f\|_{H(K)}\le 1} \|f - P_{m-1}f\|^2_{\ell_\infty(D)} = \sup_{\mathbf{x}\in D} \sum_{k\ge m} |e_k^*(\mathbf{x})|^2$$

$$= \sup_{\mathbf{x}\in D} \sum_{l=\lfloor \log_2 m\rfloor}^{\infty} \sum_{2^l\le k<2^{l+1}} |\sigma_k \eta_k^*(\mathbf{x})|^2$$

For all $2^l \le k < 2^{l+1}$ it holds $\sigma_k^2 \le \frac{1}{2^{l-1}} \sum_{2^{l-1}\le j<2^l} \sigma_j^2$

$$\le \sum_{l=\lfloor \log_2 m\rfloor}^{\infty} \frac{N_{K,\varrho_D}(2^{l+1})}{2^{l-1}} \sum_{2^{l-1}\le j<2^l} \sigma_j^2 \le \cdots \le 2 \sum_{k\ge\lfloor m/2\rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}$$