Sampling recovery. Lecture 1.
Recovery in the L_p norms.

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Trigonometric polynomials. Dirichlet kernel.

Functions of the form

$$t(x) = \sum_{|k|\le n} c_k e^{ikx} = a_0/2 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

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The Dirichlet kernel of order n

$$\mathcal{D}_n(x) := \sum_{|k|\le n} e^{ikx} = e^{-inx}(e^{i(2n+1)x} - 1)(e^{ix} - 1)^{-1}$$

$$= \big(\sin(n+1/2)x\big) \; / \; \sin(x/2).$$

Denote

$$x^j := 2\pi j/(2n+1), \qquad j = 0, 1, ..., 2n.$$

Clearly, the points x^j, $j = 1, \dots, 2n$, are zeros of the Dirichlet kernel $\mathcal{D}_n$ on $[0, 2\pi]$.

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Clearly, the points x^j, $j = 1, \dots, 2n$, are zeros of the Dirichlet kernel $\mathcal{D}_n$ on $[0, 2\pi]$. Consequently, for any continuous f

$$I_n(f)(x) := (2n+1)^{-1} \sum_{j=0}^{2n} f(x^j)\mathcal{D}_n(x - x^j)$$

interpolates f at points x^j: $I_n(f)(x^j) = f(x^j)$, $j = 0, 1, ..., 2n$.

Error of interpolation

It is easy to check that for any $t \in \mathcal{T}(n)$ we have $I_n(t) = t$. Using this and the inequality

$$\big|\mathcal{D}_n(x)\big| \le \min\big(2n+1, \pi/|x|\big), \qquad |x| \le \pi,$$

we obtain

$$\|f - I_n(f)\|_\infty \le C\ln(n+1)E_n(f)_\infty,$$

where $E_n(f)_p$ is the best approximation of f in the L_p norm by polynomials from $\mathcal{T}(n)$.

The de la Vallée Poussin kernels

$$\mathcal{V}_{2n}(x) := n^{-1} \sum_{k=n}^{2n-1} \mathcal{D}_k(x) = \frac{\cos nx - \cos 2nx}{n(\sin(x/2))^2}.$$

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Consider the following recovery operator

$$R_n(f) := (4n)^{-1} \sum_{j=1}^{4n} f\big(x(j)\big)\mathcal{V}_n\big(x - x(j)\big), \qquad x(j) := \pi j/(2n).$$

Properties of $R_n(f)$

It is easy to check that for any $t \in \mathcal{T}(n)$ we have $R_n(t) = t$. Using this and the above majorant we obtain

$$\|f - R_n(f)\|_\infty \leq CE_n(f)_\infty.$$

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What about error in the L_p, $p \in [1, \infty)$? Let $\varepsilon := \{\epsilon_k\}_{k=0}^\infty$ be a non-increasing sequence of non-negative numbers. Define

$$E(\varepsilon, p) := \{f \in \mathcal{C} : \; E_k(f)_p \le \epsilon_k, \; k = 0, 1, \dots\}.$$

Error of recovery

Theorem (VT, 1985)

Assume that a sequence ε *satisfies the conditions: for all* $s = 0, 1, \dots$ *we have*

$$\sum_{\nu=s+1}^{\infty} \epsilon_{2^\nu} \le B\epsilon_{2^s}, \qquad \epsilon_s \le D\epsilon_{2s}.$$

Then for $p \in [1, \infty)$

$$\sup_{f \in E(\varepsilon,p)} \|f - R_n(f)\|_p \asymp \sum_{\nu=0}^{\infty} 2^{\nu/p} \epsilon_{n2^\nu}.$$

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This theorem for $1 \le p \le 2$ was proved in VT, 1985. A similar proof works for other p.

Norms of operators

In the case of space $\mathcal{C}$ ($p = \infty$) we have

$$\|R_n\|_{\mathcal{C}\to\mathcal{C}} \le C.$$

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This allows us to obtain the inequality $\|f - R_n(f)\|_\infty \le CE_n(f)_\infty$. Operators R_n are not defined on L_p, when $p < \infty$. What to do? Historically, the first idea was to consider the operator R_nJ_r where

$$J_r(f)(x) := (2\pi)^{-1}\int_{\mathbb{T}} f(x-y)F_r(y)dy,$$

$$F_r(y) := 1 + \sum_{k=1}^{\infty} k^{-r}\cos(ky - r\pi/2).$$

It was proved in VT, 1985 that for $r > 1/p$ we have (I is the identity operator)

$$\|I - R_nJ_r\|_{L_p\to L_p} \le C(r,p)n^{-r}.$$

The following inequalities turns out to be more convenient. Denote

$$V_s(f)(x) := (2\pi)^{-1}\int_{\mathbb{T}} f(x-y)\mathcal{V}_s(y)dy.$$

Then (VT, 1993) we have for $s \ge n$

$$\|R_nV_s\|_{L_p\to L_p} \le C(s/n)^{1/p}, \quad 1\le p\le\infty$$

and

$$\|I_nV_s\|_{L_p\to L_p} \le C(p)(s/n)^{1/p}, \quad 1< p<\infty.$$

Optimal recovery

For a fixed m and a set of points $\xi := \{\xi^j\}_{j=1}^m \subset \Omega$, let Φ_ξ be a linear operator from $\mathbb{C}^m$ into $L_p(\Omega,\mu)$. Denote for a class $\mathbf{F}$ (usually, centrally symmetric and compact subset of $L_p(\Omega,\mu)$)

$$\varrho_m(\mathbf{F}, L_p) := \inf_{\text{linear}\,\Phi_\xi;\,\xi} \sup_{f\in\mathbf{F}} \|f - \Phi_\xi(f(\xi^1),\dots,f(\xi^m))\|_p.$$

The above described recovery procedure is a linear procedure.

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The above described recovery procedure is a linear procedure. The following modification of the above recovery procedure is also of interest. We now allow any mapping $\Phi_\xi : \mathbb{C}^m \to X_N \subset L_p(\Omega,\mu)$ where X_N is a linear subspace of dimension $N \le m$ and define

$$\varrho_m^*(\mathbf{F}, L_p) := \inf_{\Phi_\xi;\xi;X_N,N\le m} \sup_{f\in\mathbf{F}} \|f - \Phi_\xi(f(\xi^1),\dots,f(\xi^m))\|_p.$$

In both of the above cases we build an approximant, which comes from a linear subspace of dimension at most m.

Univariate smoothness classes

Define

$$W_q^r := \{f \,:\, f = J_r(\varphi),\ \|\varphi\|_q \le 1\}.$$

Univariate smoothness classes

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$$W^r_q := \{f : f = J_r(\varphi),\ \|\varphi\|_q \le 1\}.$$

Theorem (VT, 1993)

Let $1 \le q, p \le \infty$ *and* $r > 1/q$. *Then*

$$\varrho_{4m}(W^r_q, L_p) \asymp \sup_{f\in W^r_q} \|f - R_m(f)\|_p \asymp m^{-r+(1/q-1/p)_+}.$$

In the case $1 < p < \infty$ *the above estimates are valid for the operator* I_m *instead of the operator* R_m.

For $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d_+$) define

$$J_{\mathbf{r}}(f)(\mathbf{x}) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(\mathbf{x} - \mathbf{y}) F_{\mathbf{r}}(\mathbf{y}) d\mathbf{y},$$

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and

$$\mathbf{W}^{\mathbf{r}}_q := \{f \,:\, f = J_{\mathbf{r}}(\varphi),\ \|\varphi\|_q \le 1\}.$$

Recovery operators

Let for $i = 1, \dots, d$ operator R^i_n be the operator R_n acting with respect to the variable x_i. Denote

$$\Delta^i_s := R^i_{2^s} - R^i_{2^{s-1}}, \quad R_{1/2} = 0,$$

and for $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d_0$

$$\Delta_{\mathbf{s}} := \prod_{i=1}^{d} \Delta^i_{s_i}.$$

Consider the recovery operator (Smolyak operator)

$$T_n := \sum_{\mathbf{s}:\|\mathbf{s}\|_1 \le n} \Delta_{\mathbf{s}}.$$

Operator T_n uses m function values with $m \ll \sum_{k=1}^n 2^k k^{d-1} \ll 2^n n^{d-1}$.

The following bound was obtained by S. Smolyak in 1960. Let $\mathbf{r} = (r, \dots, r)$. In this case write $\mathbf{W}^{\mathbf{r}}_q = \mathbf{W}^r_q$. Then

$$\sup_{f \in \mathbf{W}^r_\infty} \|f - T_n\|_\infty \ll 2^{-rn} n^{d-1}, \quad r > 0.$$

It was extended to the case $p < \infty$ in VT, 1985:

$$\sup_{f \in \mathbf{W}^r_p} \|f - T_n\|_p \ll 2^{-rn} n^{d-1}, \quad r > 1/p.$$

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Open problem. Find the right order of the optimal sampling recovery $\varrho_m(\mathbf{W}^r_p, L_p)$ in case $1 \le p \le \infty$ and $r > 1/p$.

We have (VT, 1993)

$$\varrho_m(\mathbf{W}^r_2)_\infty \asymp m^{-r+1/2}(\log m)^{r(d-1)}, \quad r > 1/2.$$

The order of optimal recovery is provided by the Smolyak operator T_n.

Further results

We have (VT, 1993)

$$\varrho_m(\mathbf{W}^r_2)_\infty \asymp m^{-r+1/2}(\log m)^{r(d-1)}, \quad r > 1/2.$$

The order of optimal recovery is provided by the Smolyak operator T_n. Also we know (VT, 1993)

$$\sup_{f\in\mathbf{W}^r_q} \|f - T_n(f)\|_\infty \asymp 2^{-(r-1/q)n} n^{(d-1)(1-1/q)}.$$

Useful inequalities

For $\mathbf{s} \in \mathbb{Z}_+^d$ define

$$\rho(\mathbf{s}) := \{\mathbf{k} \in \mathbb{Z}^d : [2^{s_j-1}] \le |k_j| < 2^{s_j}, \quad j = 1, \dots, d\}$$

where $[x]$ denotes the integer part of x and

$$\delta_{\mathbf{s}}(f)(\mathbf{x}) := \sum_{\mathbf{k} \in \rho(\mathbf{s})} \hat{f}(\mathbf{k}) e^{i(\mathbf{k},\mathbf{x})}.$$

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Let an array $\varepsilon = \{\varepsilon_{\mathbf{s}}\}$ be given, where $\varepsilon_{\mathbf{s}} \ge 0$, $\mathbf{s} = (s_1, \dots, s_d)$, and s_j are nonnegative integers, $j = 1, \dots, d$.

Notations

We denote by $G(\varepsilon, q)$ and $F(\varepsilon, q)$ the following sets of functions $(1 \le q \le \infty)$:

$$G(\varepsilon, q) := \left\{ f \in L_q : \left\| \delta_{\mathbf{s}}(f) \right\|_q \le \varepsilon_{\mathbf{s}} \qquad \text{for all } \mathbf{s} \right\},$$

$$F(\varepsilon, q) := \left\{ f \in L_q : \left\| \delta_{\mathbf{s}}(f) \right\|_q \ge \varepsilon_{\mathbf{s}} \qquad \text{for all } \mathbf{s} \right\}.$$

Estimating $\|f\|_p$

Theorem (VT, 1986)

The following relations hold:

$$\sup_{f\in G(\varepsilon,q)} \|f\|_p \asymp \left(\sum_{\mathbf{s}} \varepsilon_{\mathbf{s}}^p 2^{\|\mathbf{s}\|_1(p/q-1)}\right)^{1/p}, \qquad 1\le q<p<\infty;$$

$$\inf_{f\in F(\varepsilon,q)} \|f\|_p \asymp \left(\sum_{\mathbf{s}} \varepsilon_{\mathbf{s}}^p 2^{\|\mathbf{s}\|_1(p/q-1)}\right)^{1/p}, \qquad 1<p<q\le\infty, \tag{1}$$

with constants independent of ε.

Remark

Remark (Dinh Zung, 1991; VT, 1993)

In the proof of first relation of Theorem (VT, 1986) we used only the property $\delta_{\mathbf{s}}(f) \in \mathcal{T}(2^{\mathbf{s}}, d)$*. That is, if*

$$f = \sum_{\mathbf{s}} t_{\mathbf{s}}, \qquad t_{\mathbf{s}} \in \mathcal{T}(2^{\mathbf{s}}, d),$$

then for $1 \leq q < p < \infty$,

$$\|f\|_p \leq C(q,p,d) \left(\sum_{\mathbf{s}} \|t_{\mathbf{s}}\|_q^p 2^{\|\mathbf{s}\|_1 (p/q-1)} \right)^{1/p}.$$

H classes

For $s \in \mathbb{N}_0$ define the univariate operators

$$A_s := V_{2^s} - V_{2^{s-1}}, \quad V_{1/2} = 0$$

and for $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}_0^d$

$$A_{\mathbf{s}} := \prod_{i=1}^{d} A^i_{s_i}.$$

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$$\mathbf{H}^r_p := \{f \,:\, \|A_{\mathbf{s}}(f)\|_p \le 2^{-r\|\mathbf{s}\|_1}\}.$$

Recovery of $\mathbf{H}$ classes

Theorem (VT, 1985)

Let $1 \le p \le \infty$ *and* $r > 1/p$. *Then we have for* $f \in \mathbf{H}^r_p$

$$\|\Delta_{\mathbf{s}}(f)\|_p \ll 2^{-r\|\mathbf{s}\|_1} \quad \textit{and} \quad \|f - T_n(f)\|_p \ll 2^{-rn} n^{d-1}.$$

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The above Theorem (VT, 1985), Theorem (VT,1986) and remark to it imply:

Theorem (Dinh Zung, 1991; VT, 1993)

For any $f \in \mathbf{H}^r_q$, $1 \le q < p < \infty$, $r > 1/q$

$$\|f - T_n(f)\|_p \ll 2^{-n(r-\beta)} n^{(d-1)/p}, \quad \beta := 1/q - 1/p.$$

One more right order result

It easily follows from the definition of $\varrho_m(\mathbf{F})_p$ that $\varrho_m(\mathbf{F})_p \ge d_m(\mathbf{F}, L_p)$, where $d_m(\mathbf{F}, L_p)$ is the Kolmogorov width. The upper bound from Theorem (VT, 1985) and the lower bound for the Kolmogorov width from VT, 1998: for $d = 2$

$$d_m(\mathbf{H}^r_\infty, L_\infty) \asymp m^{-r}(\log m)^{r+1}$$

imply for $d = 2$

$$\varrho_m(\mathbf{H}^r_\infty)_\infty \asymp m^{-r}(\log m)^{r+1}.$$

Partial sums

For $N \in \mathbb{N}$ define the hyperbolic cross

$$\Gamma(N) := \{\mathbf{k} \in \mathbb{Z}^d \ : \ \prod_{j=1}^{d} \max(|k_j|, 1) \le N\}$$

and the corresponding Dirichlet kernel

$$\mathcal{D}_N(\mathbf{x}) := \sum_{\mathbf{k} \in \Gamma(N)} e^{i(\mathbf{k},\mathbf{x})}.$$

Consider the hyperbolic cross partial sums

$$S_N(f, \mathbf{x}) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(\mathbf{y}) \mathcal{D}_N(\mathbf{x} - \mathbf{y}) d\mathbf{y}.$$

Approximation

It is known that

$$\sup_{f\in \mathbf{W}^r_2} \|f - S_N(f)\|_2 \asymp d_{|\Gamma(N)|}(\mathbf{W}^r_2, L_2) \asymp N^{-r}.$$

For a point set $\xi(m) = \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$ consider a discretization of the convolution operator S_N

$$S_N(f,\xi(m),\mathbf{x}) := \frac{1}{m}\sum_{\nu=1}^m f(\xi^\nu)\mathcal{D}_N(\mathbf{x}-\xi^\nu).$$

How many points do we need to guarantee

$$\sup_{f\in \mathbf{W}^r_2} \|f - S_N(f,\xi(m))\|_2 \asymp d_{|\Gamma(N)|}(\mathbf{W}^r_2, L_2) \asymp N^{-r}? \qquad (2)$$

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It is proved in VT, 1986 that it is sufficient to take $m \asymp N^2(\log N)^{d-1}$ for (2) to hold. The proof uses number theoretical constructions.

Thank you!