Tchebycheffian B-splines in Isogeometric Analysis

from geometric modelling to numerical simulation

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joint work with
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Geometric modelling

- methods/algorithms to construct, represent,
    - curves
    - surfaces
    - volumes
    - ...
- of interest in:
    - CAD/CAM
    - robotics
    - scientific imaging and visualization
    - ...

B-splines

- B-splines are the $\boxed{\text{mathematical}}$ core of any CAD system
- B-splines are a special form to represent any piecewise polynomial (p.p.) function/curve/surface

Outline

B-splines: " knots make B-splines" (C. de Boor)

- Given a set of knots $\Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1}\}$
- B-splines are defined recursively

$$B_{i,0,\Xi}(t) := \begin{cases} 1 & \text{if } t \in [\xi_i, \xi_{i+1}) \\ 0 & \text{elsewhere} \end{cases}$$

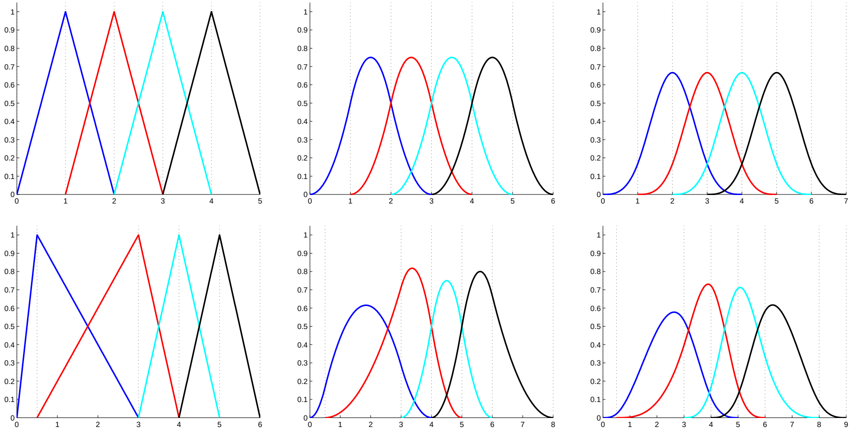
$$B_{i,p,\Xi}(t) := \frac{t-\xi_i}{\xi_{i+p}-\xi_i} B_{i,p-1,\Xi}(t) + \frac{\xi_{i+1+p}-t}{\xi_{i+1+p}-\xi_{i+1}} B_{i+1,p-1,\Xi}(t), \quad p \geq 2$$

$\boxed{B_{i,p,\Xi} : \ i\text{-th B-spline, of degree } p, \text{ with knots } \Xi}$

[de Boor], [Popoviciu and Chakalov, 1930]

B-splines: "knots make B-splines" (C. de Boor)

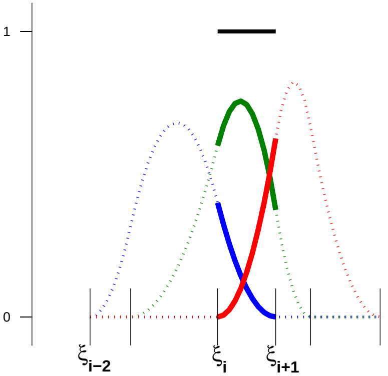
- Given a set of knots $\Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1}\}$
  - B-splines are piecewise polynomial functions
  - B-splines have minimum support: $p+1$ knot intervals

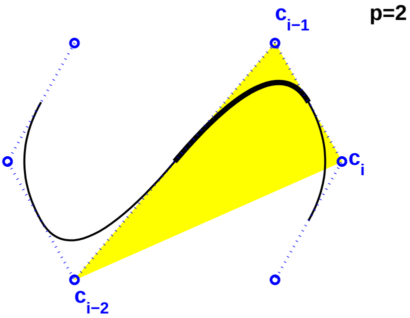


$p = 1$ $p = 2$ $p = 3$

B-splines: "knots make B-splines" (C. de Boor)

- Given a set of knots $\Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1}\}$
  - B-splines have minimum support
  - B-splines are all non negative and form a partition of unity





$\sum_i \mathbf{c}_i B_{i,p,\Xi}$

B-splines

- Why are B-splines so popular?
    - they enjoy several nice properties clearly deduced from the knots
    - there exist efficient and stable algorithms for their evaluation/manipulation/refinement
    - they are the best way to represent p.p. from the geometrical point of view (optimal totally positive basis)

B-splines as an approximation tool

- condition number

$$K_p^{-1}\|\mathbf{c}\|_\infty \leq \left\| \sum_j c_j B_{j,p,\Xi} \right\|_\infty \leq \|\mathbf{c}\|_\infty$$

  - $0 < K_p$ only depends on p
  - extensions to any L_q norm

- approximation power

  $f \in W_q^{\ell+1}([a,b])$, $0 \leq \ell \leq p$, $\Rightarrow \exists s_p \in \langle B_{1,p,\Xi}, \dots, B_{n,p,\Xi} \rangle$

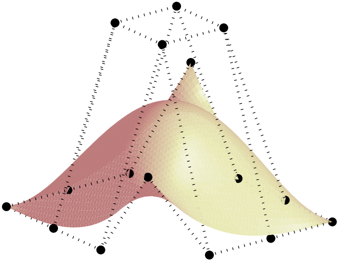
$$\|D^r(f - s_p)\|_{L_q([a,b])} \leq K h_\Xi^{\ell+1-r} \|D^{\ell+1} f\|_{L_q([a,b])}, \quad 0 \leq r \leq \ell$$

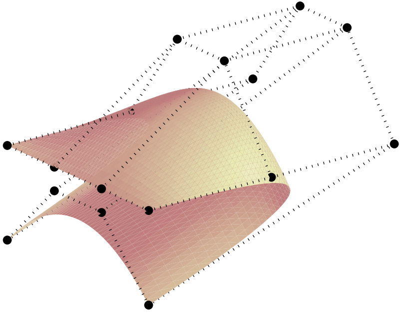
  - $h_\Xi := \max \ \xi_{i+1} - \xi_i$
  - K is independent of Ξ but depends on p, on the smoothness of the spline space, and on ℓ;
  - explicit expression for K in L_2 by using Kolmogorov $n-$*widths* [Sande, Manni, Speleers, 2019], [Sande, Manni, Speleers, 2020]

Tensor-product B-splines

$$\Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p}\}, \ \Upsilon := \{\upsilon_1 \leq \upsilon_2 \leq \cdots \leq \upsilon_{m+q}\}$$

$$\mathcal{S}(u,v) = \sum_{i=1,j=1}^{n,m} \mathbf{c}_{i,j} B_{i,p,\Xi}(u) B_{i,q,\Upsilon}(v)$$





B-splines → NURBS

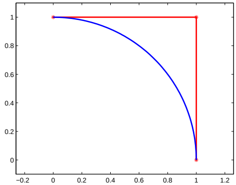
- no hope for exact representations of conic sections (circles,...)
  Ex. unit circle $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$

$$\Downarrow$$

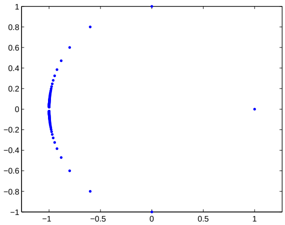
- NURBS: Non Uniform Rational B-Splines
  $\{B_{i,p,\Xi}(t),\ i=1,\cdots,n\},\ W := \{w_i \geq 0,\ i=1,\cdots\},$ weights

$$\boxed{R_{i,p,\Xi,W}(t) := \frac{w_i B_{i,p,\Xi}(t)}{\sum_{j=1}^{n} w_j B_{j,p,\Xi}(t)}}$$

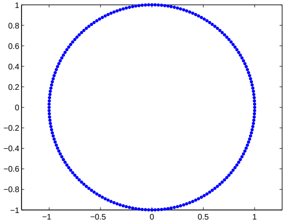
- NURBS: projective transformation of B-splines
  - positivity, TP basis
  - p. of unity
  - compact support
  - smoothness
  - exact representation of (segments of) conic sections
  - ...



B-splines/NURBS: so far so good ... but



$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$: 200 "equispaced" points



$(\sin(t), \cos(t))$: 200 equispaced points

only straight lines have rational repres. w.r.t. arc length

Drawbacks of the rational model

- Geometry
  - rational curves require additional parameters (weights) whose selection is often not clear
  - the rational model cannot encompass transcendental curves: many of them (helix, cicloid, ...) are of interest in applications
  - parametrization of conic sections does not correspond to natural arc-length parametrization: unevenly spaced points
- Analysis
  - the derivative of a degree-p integral curve is of degree $p-1$: the derivative of a degree-p rational curve is of degree $2p$...
  - exact integration of rational curves is hard and requires (whenever possible) non rational forms

Alternatives: reproducing conic sections, cycloids ....

GEOMETRY

$$< 1, t, \dots, t^{p-2}, t^{p-1}, t^{p} >$$
$$< 1, t, \dots, t^{p-2}, e^{\alpha t}, e^{-\alpha t} >,$$
$$< 1, t, \dots, t^{p-2}, \cos \alpha t, \sin \alpha t >,$$
$$< 1, t, \dots, t^{p-\ell}, e^{\beta_1 t}, \dots, e^{\beta_\ell t} >, \quad \ell \leq p$$

ANALYSIS

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} < 1, t, \dots, t^{p-2}, t^{p-1}, t^{p} > &= < 1, t, \dots, t^{p-3}, t^{p-2}, t^{p-1} > \\
\frac{\mathrm{d}}{\mathrm{d}t} < 1, t, \dots, t^{p-2}, e^{\alpha t}, e^{-\alpha t} > &= < 1, t, \dots, t^{p-3}, e^{\alpha t}, e^{-\alpha t} > \\
\frac{\mathrm{d}}{\mathrm{d}t} < 1, t, \dots, t^{p-2}, \cos \alpha t, \sin \alpha t > &= < 1, t, \dots, t^{p-3}, \cos \alpha t, \sin \alpha t > \\
\frac{\mathrm{d}}{\mathrm{d}t} < 1, t, \dots, t^{p-\ell}, e^{\beta_1 t}, \dots, e^{\beta_\ell t} > &= < 1, t, \dots, t^{p-\ell-1}, e^{\beta_1 t}, \dots, e^{\beta_\ell t} >,
\end{aligned}$$

NURBS $\quad R^{(p)}_{i,\Xi,W}(t) := \dfrac{w_i B^{(p)}_{i,\Xi}(t)}{\sum_k w_k B^{(p)}_{k,\Xi}(t)}, \; i = 1, \dots \qquad \dfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{NURBS} = ?$

alternatives to the rational model retaining properties of B-splines?

Beyond polynomials

- $\mathbb{P}_p := < 1, t, \ldots, t^{p-2}, t^{p-1}, t^p >$
- $\mathbb{P}_p^{u,v} := < 1, t, \ldots, t^{p-2}, u(t), v(t) >, \ p \geq 2 \quad t \in [a, b]$
- $\mathbb{T}_p := < 1, t, \ldots, t^{p-\ell}, e^{\beta_1 t}, \ldots, e^{\beta_\ell t} >$

$\mathbb{P}_p^{u,v}$, $\mathbb{T}_p$: extended Tchebycheff space on $[a, b]$

any non trivial element has at most p zeros in $[a, b]$ (counting multiplicity)

- trigonometric functions $< 1, t, \ldots, t^{p-2}, \cos \alpha t, \sin \alpha t >$
- exponential functions $< 1, t, \ldots, t^{p-2}, e^{\alpha t}, e^{-\alpha t} >$
- kernel (null space)of differential operator of order p with real (constant) coefficients
- ....

Alternatives to the rational model

- rational model: $\mathbb{P}_p \to$ B-splines $\to$ NURBS

- alternative:

$$\mathbb{P}_p :=< 1, t, \ldots, t^{p-2}, t^{p-1}, t^p >$$

$$\downarrow$$

$$\mathbb{P}_p^{u,v} :=< 1, t, \ldots, t^{p-2}, u(t), v(t) >$$

$$\mathbb{T}_p :=< 1, t, \ldots, t^{p-\ell}, e^{\beta_1 t}, \ldots, e^{\beta_\ell t} >$$

- construct/analyse spline spaces with sections in $\mathbb{P}_p^{u,v}, \mathbb{T}_p$ with suitable bases (analogous to B-splines)

$\boxed{\mathbb{P}_p^{u,v}\text{: Generalized B-splines (GB), } \mathbb{T}_p\text{: Tchebycheffian B-splines (TB)}}$

[Jerome, Schumaker: 1976], [Lyche: 1985], [Schumaker: 1993],[Koch, Lyche: 1993],
[Kvasov, Sattayatham: 1999],
[Costantini: 2000], [Costantini, Lyche, Manni: 2005], [Costantini, Manni: 2006],
[Wang, Fang: 2008],
[Mazure: 2011], [Mazure, 2012], Mazure[2015], [Mazure, 2016] ...
[Lyche, Manni, Speleers, 2019]
[Beccari, Casciola, Mazure,2019]
...

Generalized B-splines

- Given a set of knots $\Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1}\}$

$$\mathbb{P}_p^{u_i,v_i} :=< 1, t, \ldots, t^{p-2}, u_i(t), v_i(t) >,$$

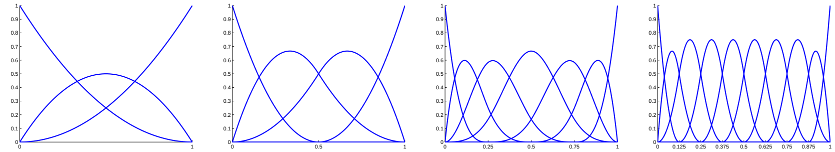
- Generalized B-spline basis functions are defined recursively

$$\boxed{\hat{B}_{i,p,\Xi} : \ i\text{-th Generalized B-spline, of degree } p, \text{ with knots } \Xi}$$

  - they have minimum support
  - they are a basis for piecewise $\mathbb{P}_p^{u_i,v_i}$
  - they are all non negative and form a partition of unity

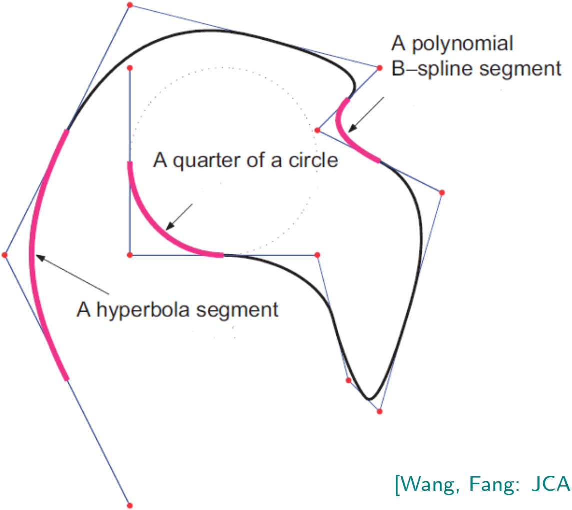
EXP$_2$ $\mathbb{P}_2^{u,v} :=< 1, e^{\alpha t}, e^{-\alpha t} >$ $\alpha \to 0$: B-splines

TRIG$_3$ $\mathbb{P}_3^{u,v} :=< 1, t, \cos \alpha t, \sin \alpha t >$ $\alpha \to 0$: B-splines



Generalized B-splines: design

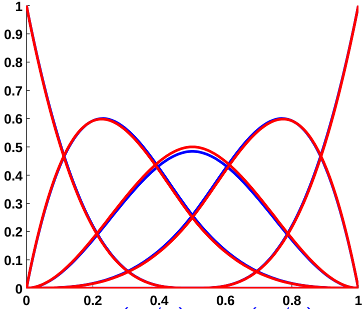
- different spaces can be mixed



[Wang, Fang: JCAM 2008]

Generalized B-splines: approximation power

- trig/exp: same approximation properties as B-splines

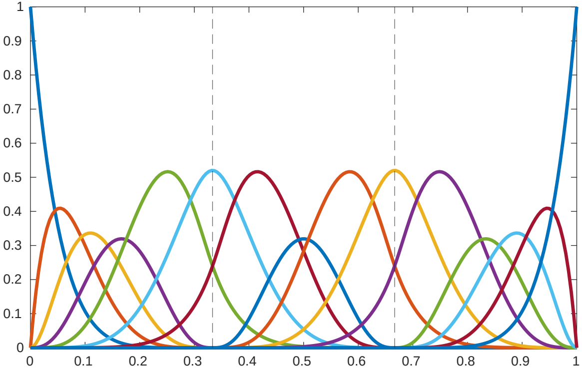


$p = 3$ $\quad < 1, t, \cos(\pi/2)t, \sin(\pi/2)t >$ $\quad$ B-splines

Tchebycheffian B-splines

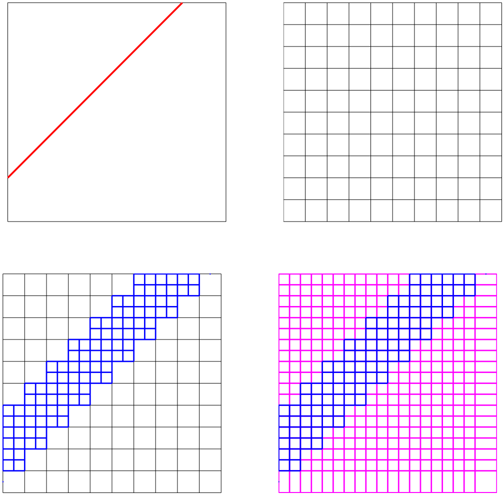
- similar properties as GB

$$\mathbb{T}_p := < 1, t, t^2, e^{10t}, e^{-10t}, \cos(t\pi), \sin(t\pi) >$$



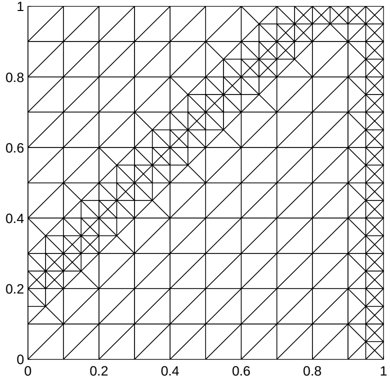
Tensor product structures:DRAWBACKS

- multivariate setting: Tensor product



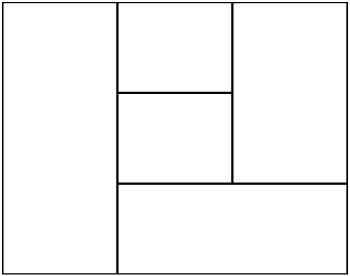
Tensor product structures: DRAWBACKS

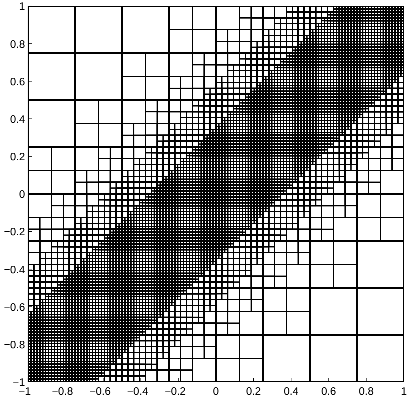
- multivariate setting: Tensor product



Tensor product structures: DRAWBACKS

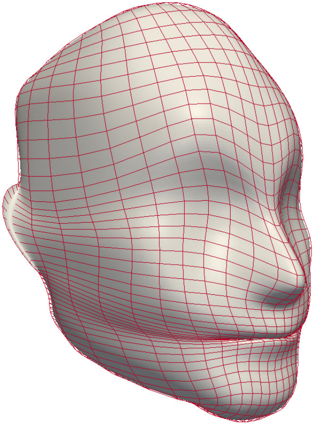
- multivariate setting: Tensor product

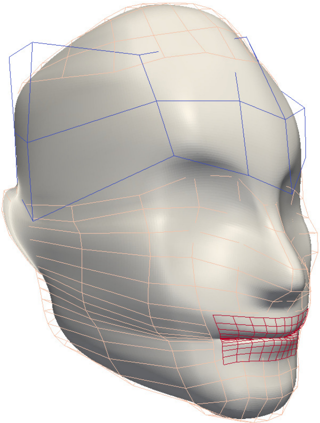




Tensor product structures: DRAWBACKS

- multivariate setting: Tensor product
- ☹tensor-product structure NO efficient local refinement
- ☺local tensor-product structure: T-meshes





Local tensor product structures

- ☹tensor-product structure NO efficient local refinement
  Alternatives (polynomial B-splines):
  - T-splines
  - Splines over T-meshes
  - LR splines
  - Hierarchical bases

  [Sederberg, Zheng, Bakenov, Nasri: 2003], [Schumaker, Wang: 2009],[Dokken, Lyche, Pettersen: 2013], [Giannelli, Juttler, Speleers: 2012], ...

- ☹tensor-product GB/TB-splines NO efficient local refinement.
  Alternatives:
  - Generalized T-splines
  - Generalized Splines over T-meshes
  - LR GB-splines
  - Hierarchical bases for GB-splines

  [Bracco, Cho: 2014], [Bracco, Lyche, Manni, Roman, Spleeers: 2015, 2016],

  [Bracco, Lyche, Manni, Spleeers, 2019]

$\boxed{\text{dimension, bases, stability, approximation power}}$

B-splines and their generalizations in simulation

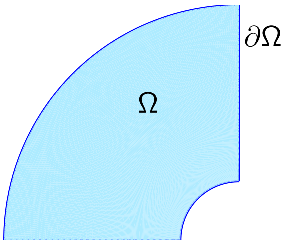
Isogeometric Analysis (IgA): paradigm for PDEs

- Isogeometric Analysis (IgA) is a unifying framework for
  - Computed aided design (CAD)
  - Finite element analysis (FEA)

[Cottrell, Hughes, Bazilevs; CMAME 2005]

The problem

- $$\begin{cases} \mathcal{L}u = & f, \quad in\ \Omega \\ \Gamma u = & g \quad on\ \partial\Omega \end{cases}$$



$u \in \mathcal{V}$

- approximating space

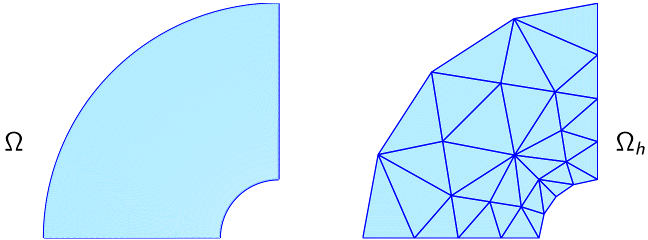
$$\mathcal{V}_h := < \phi_1, \phi_2, \ldots, \phi_{n_h} > \subset \mathcal{V}$$

- approximate solution $u_h \in \mathcal{V}_h$
  to be selected by a suitable approximating strategy
  (Galerkin, collocation...)

Isoparametric approach: FEM

$$\mathcal{V}_h := \{v_h \in C^0(\Omega_h) : v_{h|K} \in \mathbb{P}_p, \ \forall K \in T_h\}$$

T_h : triangulation of a (polygonal) approximation (Ω_h) of Ω



To improve accuracy:

- grid refinement, decresing h ($h-$refinements)
- increase the degree p of the “elements” ($p-$refinements)

Engineering Analysis Process

- Physical domain is an output of CAD systems
- CAD geometry is replaced by FEM geometry (*mesh*)
- Mesh generation: more than 80% of overall analysis time
- Mesh refinement requires continuous interaction with CAD geometry

Moreover...

- The mesh is an approximate geometry: many problems (thin shell structures, boundary layer in fluids) are very sensitive to geometric imperfections

$\Downarrow$

Develop an analysis framework based on functions capable of exactly/better representing geometry

$\Downarrow$

$\boxed{\text{Isogeometric Analysis}}$

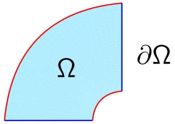
"*the solution space for dependent variables is represented in terms of the same functions which represent the geometry*"

[Cottrell, Hughes, Bazilevs; CMAME 2005]

The problem: Galerkin

- Second order (elliptic) partial differential equation (PDE),

$$\mathcal{L}u = \begin{cases} Lu = & \mathrm{f}, \quad in\ \Omega \\ \Gamma u = & \mathrm{g} \quad on\ \partial\Omega \end{cases}$$



- weak formulation:

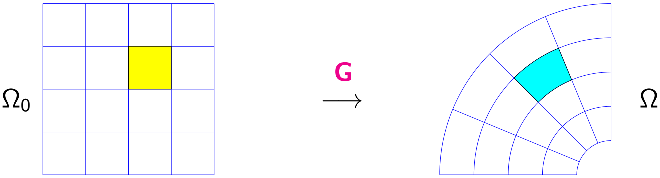
$$\text{Find} \quad u \in \mathcal{V}, \ \text{ such that } \ a(u,v) = F(v), \ \ \forall\ v \in \mathcal{V}$$

$a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ bilinear form depending on L
$F : \mathcal{V} \to \mathbb{R}$ linear form depending on f and g.

Example: $\begin{cases} -\Delta u = & \mathrm{f}, \quad in\ \Omega \\ u = & 0 \quad on\ \partial\Omega \end{cases}$

Isogeometric Analysis (IgA)

- $\Omega_0 := [0,1]^2$: parametric domain, Ω : physical domain
- global geometry function $\mathbf{G} : \Omega_0 \to \Omega$:
  $\mathbf{G}(\xi) = \sum_{i=1}^{n_h} B_i(\xi)\mathbf{c}_i, \qquad \{B_1, \cdots, B_{n_h}\}$: basis



$\xi \in \Omega_0$ $\qquad$ $x(\xi) \in \Omega$

$$u_h(x) = \sum_{i=1}^{n_h} \underbrace{B_i \circ \mathbf{G}^{-1}(x)}_{\text{basis functions } \phi_i} q_i, \quad x \in \Omega,$$

IgA Galerkin based on NURBS

$\boxed{\text{basis functions in } \Omega_0\text{: tensor-product NURBS}}$

IgA based on NURBS: benefits

- ☺ more accurate modelling of complex geometries/exact representations of common engineering shapes (conic sections ...)
- ☺ geometrically exact: no matter how coarse is the discretization
  ☺ $\Rightarrow$ simplified mesh refinement by eliminating the need for communication with the CAD geometry

IgA based on NURBS: benefits

- ☺ several efficient possibilities of refinement
- ☺ additional global smoothness is regarded as beneficial [Cottrell, Hughes, Bazilevs; CMAME 2005]
- ☺ existence of efficient and stable algorithms for evaluation and representation (B-spline representations)

IgA based on NURBS: benefits

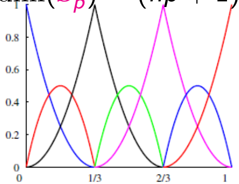
$\boxed{\text{reduction of degrees of freedom – same accuracy}}$

dimension 1, n elements, degree p

dimension d, n^d elements, degree p

$$\dim(\mathbb{S}_p^0) = np + 1, \quad \dim(\mathbb{S}_p^{p-1}) = n + p$$

$$\dim(\mathbb{S}_p^0) = (np + 1)^d, \quad \dim(\mathbb{S}_p^{p-1}) = (n + p)^d$$



$\Xi = \{0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, 1, 1, 1\}, p = 2$

$\Xi = \{0, 0, 0, \frac{1}{3}, \frac{2}{3}, 1, 1, 1\}, p = 2$

[Cottrell, Hughes, Bazilevs; CMAME 2005]

Alternative to the rational model in IgA

" NURBS are not a requisite ingredient in isogeometric analysis. We might envision developing isogeometric procedures based on..."

[Cottrell, Hughes, Bazilevs; CMAME 2005]

tensor product GB/TB-splines: basis functions in Ω_0

Section spaces to be selected with a problem-dependent strategy

NURBS and GB/TB-splines are plug-to-plug in IgA *

GB/TB-spline based IgA $\Rightarrow$ same benefits as NURBS based IgA

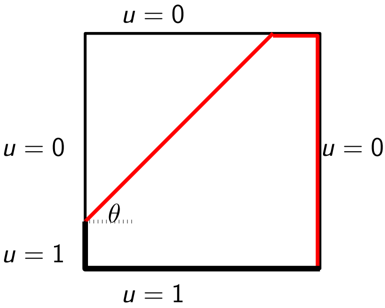
GB/TB-spline based IgA: some benefits over NURBS based IgA

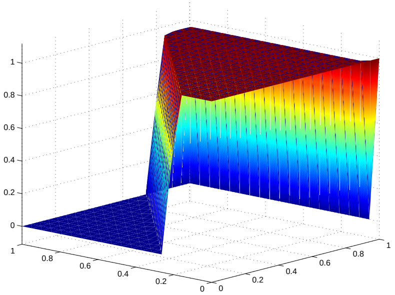
$*$ thanks to efficient evaluation algorithms based on Bézier extraction recently provided [Hiemstra, Hughes, Manni,Speleers, andToshniwal 2020], [Speleers, 2021]

GB-splines based IgA: advection

$\boxed{\text{Section spaces to be selected with a problem-dependent strategy}}$

strong gradients/thin layers $\Rightarrow$ Exp.or Variable degree B-splines





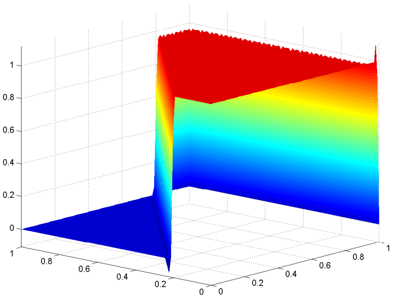
$$-\varepsilon \triangle u + \mathbf{b} \cdot \nabla u = 0, \qquad \mathbf{b} = (\cos(\theta), \sin(\theta)), \ \varepsilon = 10^{-6}$$

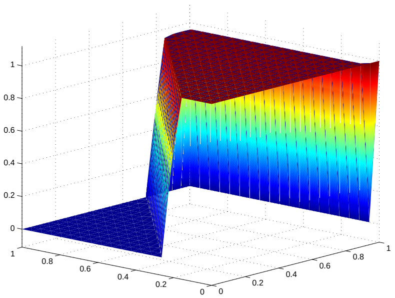
[Manni, Pelosi, Sampoli, JCAM 2011]

GB-splines based IgA: advection

[Manni, Pelosi, Sampoli, JCAM 2011]

strong gradients/thin layers $\Rightarrow$ Exp.or Variable degree B-splines





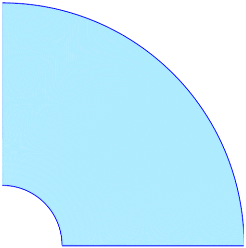
C^0 FEM $\mathbb{P}_2$ three-directional mesh 40×40 $\qquad$ C^3 quintic VD mesh 40×40

TB-splines based IgA: advection & non trivial domain

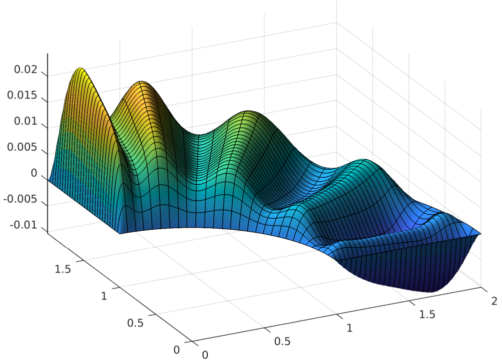
[Manni, Raval, Speleers, in preparation]

advection in tangential direction

$$-\varepsilon \triangle u + \mathbf{b} \cdot \nabla u = 0, \qquad \mathbf{b} = \left(\frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\right), \ \varepsilon = 10^{-2}$$



Physical domain



$< 1, t, \ldots, t^6 > \otimes < 1, s, s^2 >$

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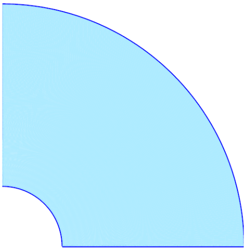
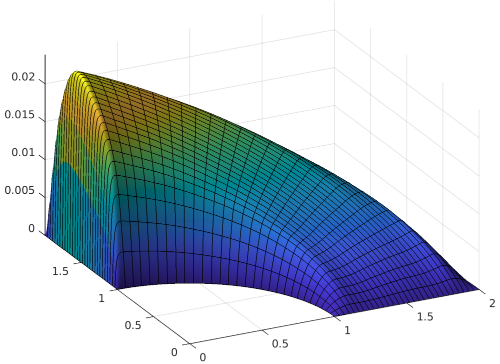


Physical domain



$< 1, t, e^{b_0 t}, e^{b_1 t}, e^{b_2 t}, \cos(t\pi/2), \sin(t\pi/2) > \otimes < 1, s, s^2 >$

b_0, b_1, b_2, automatically selected from the problem

Eigenvalue Problem and Outliers

Eigenvalue Problem: Galerkin approximation

$$\begin{cases} -u'' = \omega^2 u, & \text{in } (0,1), \\ u(0) = 0, & u(1) = 0, \end{cases}$$

$$u_j(x) := \sin(\omega_j x), \quad \omega_j := j\pi, \quad j = 1, 2, \ldots$$

Weak formulation:
find $u_j \in H_0^1$ and $\omega_j \in \mathbb{R}$, $j = 0, 1, \ldots,$ such that

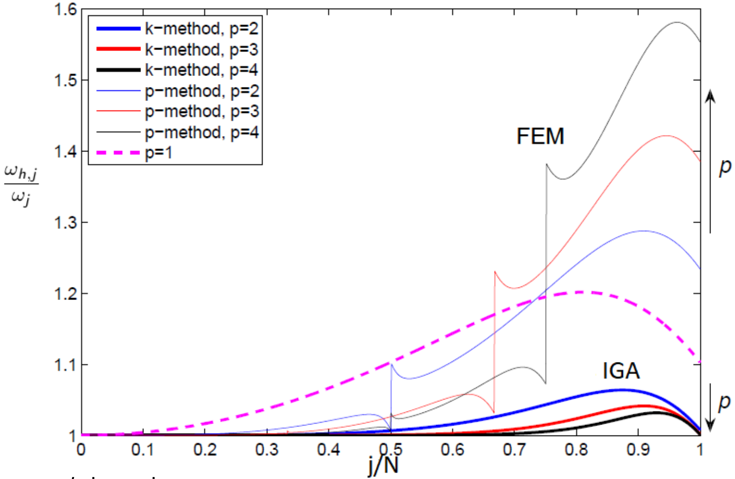
$$(\partial u_j, \partial v) = \omega_j^2 (u_j, v), \quad \forall v \in H_0^1.$$

Spline discretization:
find $u_{h,j} \in \mathbb{S}^k_{p,\boldsymbol{\tau}} \cap H_0^1$ and $\omega_{h,j} \in \mathbb{R}$, $j = 1, \ldots, N$, such that

$$(\partial u_{h,j}, \partial v) = \omega_{h,j}^2 (u_{h,j}, v), \quad \forall v \in \mathbb{S}^k_{p,\boldsymbol{\tau}} \cap H_0^1.$$

$$\mathbb{S}^k_{p,\boldsymbol{\tau}} \cap H_0^1 = \{v \in \mathbb{S}^k_{p,\boldsymbol{\tau}} : v(0) = v(1) = 0)\}, \; k \geq 0$$

N: dimension of the discretization space



- $p - k$ branches
- only a single branch converges to the true spectrum
- maximal smoothenss ($k = p - 1$) no spurious branches

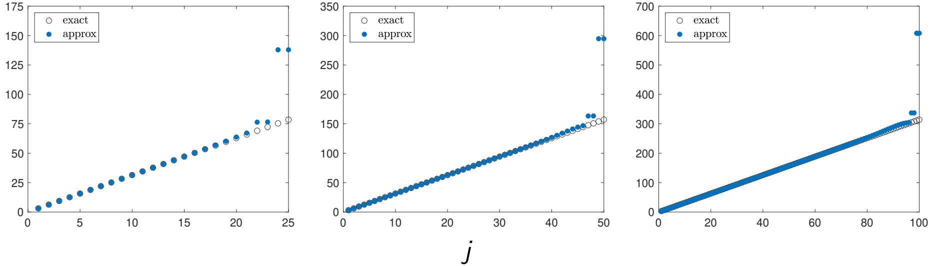
[Cottrell, Reali, Bazilevs, and Hughes, CMAME 2006]
[Garoni, Speleers, et al., Arch. Comput. Methods Eng. 2019]

Outliers

....however there is a problem for large j in $\{1, \dots, N\}$

a very small portion of the frequencies are poorly approximated and the corresponding computed values are much larger than the exact ones. These spurious values are usually referred to as outliers

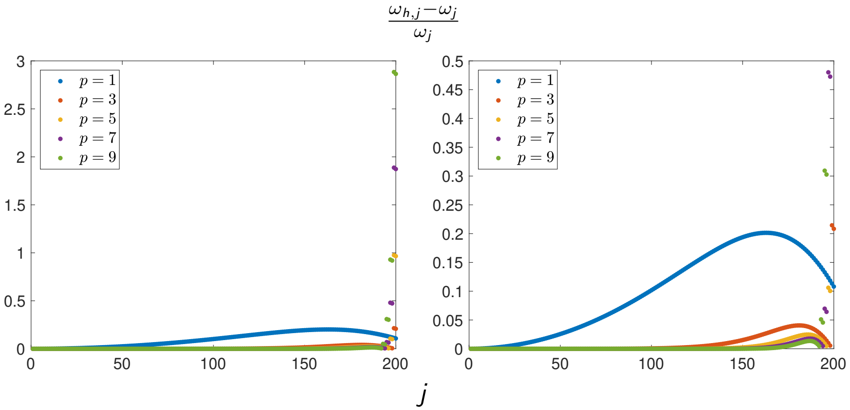
$p = 5$



[Cottrell, Reali, Bazilevs, and Hughes, CMAME 2006]
[Hughes, Reali, and Sangalli, CMAME 2008]
[Hughes, Evans, and Reali, CMAME 2014]
[Chan and Evans, CMAME 2018]
...

the number of outliers

- is independent of N (for fixed p)
- increases with p
- remains unchanged when discretizing with GB/TB



$\boldsymbol{\tau}$: $0=\tau_0<\tau_1<\cdots<\tau_{n_{\rm el}-1}<\tau_{n_{\rm el}}=1 \Rightarrow \dim(\mathbb{S}^k_{p,\boldsymbol{\tau}}\cap H^1_0)=n_{\rm el}+p-2$

Theory: the first $n_{\rm el}-1$ eigenvalues are well approximated $\Rightarrow$ at most $p-1$ outliers

[Sande, Manni, Speleers, MMMAS 2019]

Why so concerned with outliers?

Outlier-free discretizations

- provide superior description of the spectrum of the continuous operator
- in the multivariate setting, for small dofs, avoid poor approximation of a consistent portion of the spectrum;
- are beneficial in various contexts, such as an efficient selection of time-steps in (explicit) dynamics and robust treatment of wave propagation.

For a fixed degree, the challenge is to remove outliers without loss of accuracy in the approximation of all eigenfunctions.

Kolmogorov n-widths

- $A \subset L^2$: class of functions, $\mathbb{X} \subset L^2$: finite dimensional subspace

$$E(A, \mathbb{X}) := \sup_{u \in A} \inf_{v \in \mathbb{X}} \|u - v\|$$

- Kolmogorov n-width of A

$$d_n(A) := \inf_{n = \dim \mathbb{X}} E(A, \mathbb{X}).$$

- optimal subspace for A: n-dimensional subspace $\mathbb{X}$ s.t.

$$d_n(A) = E(A, \mathbb{X})$$

- Kolmogorov, Ann. of Math., 1936
- Melkman and Micchelli, Illinois J. Math.,1978
- Evans et al., CMAME 2009
- Floater and Sande, JAT 2017

Optimal spline spaces for the n-width problem

$$\mathbb{S}_{p,\boldsymbol{\tau},0} := \{s \in \mathbb{S}_{p,\boldsymbol{\tau}} : \; s^{(\alpha)}(0) = s^{(\alpha)}(1) = 0, \quad 0 \le \alpha \le p, \quad \alpha \text{ even}\}$$

- For suitable (uniform) knots $\boldsymbol{\tau}$ depending on the parity of p the n-dimensional space $\mathbb{S}_{p,\boldsymbol{\tau},0}$ is optimal (Kolmogorov $n - widths$) for

  $$\{u \in H^r : \; u^{(\alpha)}(0) = u^{(\alpha)}(1) = 0, \;\; 0 \le \alpha < r, \;\; \alpha \text{ even}, \|u^{(r)}\| \le 1\}$$

  - Floater and Sande, CA 2019
  - Floater and Sande, JCAM 2019
- the space $\mathbb{S}_{p,\boldsymbol{\tau},0}$ admits a B-spline like basis
  - Takacs and Takacs, MMMAS 2016
  - Floater and Sande, CA 2019
  - Hiemstra et al., CMAME (2021)

    we provide an explicit expression of this basis by means of linear combination of cardinal B-splines

Optimal spline spaces have no outliers

- we provide error estimates for Ritz projectors in such optimal spline subspaces $\mathbb{S}_{p,\boldsymbol{\tau},0}$
- we exploit the above estimates to show that for the considered Galerkin discretizations $\mathbb{S}_{p,\boldsymbol{\tau},0}$
- there is no loss of accuracy in the whole spectrum when compared to the full spline space
- all the first n eigenvalues/eigenfunctions are well approximated

$$\dim(\mathbb{S}_{p,\boldsymbol{\tau},0}) = n$$

[Manni, Sande, Speleers, CMAME 2022]

Optimal spline spaces have no outliers

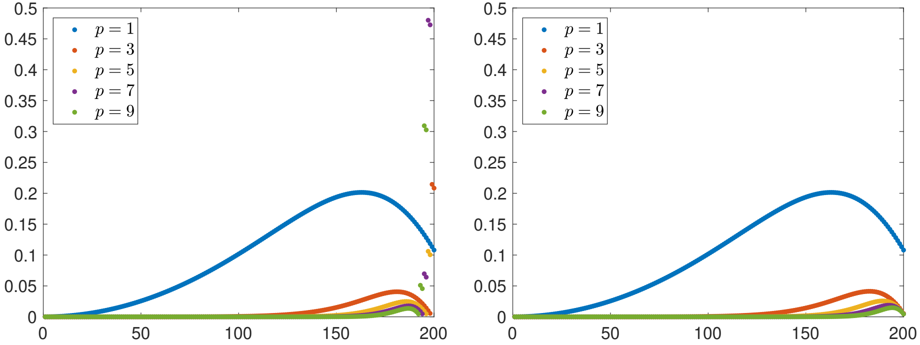


Figure: Relative frequency error for the full space and $\mathbb{S}_{p,\boldsymbol{\tau},0}$, $n = 200$.

Similar approaches have been studied numerically in
[Deng and Calo, CMAME 2021]

[Hiemstra, Hughes, Reali, Schillinger, CMAME 2021]

Concluding Message

STATE of the ART

- modelling: tensor-product B-splines/NURBS core of commercial CAD systems
- simulation: tensor-product B-splines/NURBS powerful tool in IgA
- GB/TB-splines behave similarly to NURBS, with problem-dependent improvements
  - B-splines/GB-splines/TB-splines plug-to-plug compatible in IgA
  - Galerkin
  - collocation
  - Spectral properties
  - Local refinements
  - conformal discretizations
  - BEM

Concluding Message

CHALLENGES

- despite the similarities, IgA still requires specific approaches to be competitive with FEM (mainly in 3D)
  - quadrature and matrix assembly
  - local refinement
  - fast solvers
  - ...
- despite the common root, complete interoperability between CAD systems and IgA is still far
  - trimmed/complex/multipatch geometries
  - volumetric modelling
  - microstructure modelling
  - ...

CHALLENGING APPLICATIONS

- fluido-structure interaction (cardiovascular simulations,...)
- magneto-hydrodynamic (nuclear fusion, ...)
- additive manifacturing (3D printing, ...)
- shape optimization

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Many Thanks