Perspectives and open problems for non-stationary subdivision schemes

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Joint work with: Nira Dyn, Lucia Romani, ...

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Outline

1. Subdivision schemes: what are they all about
2. Why subdivision schemes?
3. Linear subdivision and algebraic tools
   - Subdivision for curves
   - Subdivision for surfaces

Subdivision schemes: short history

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ELEMENTE DER MATHEMATIK
Revue de mathématiques élémentaires – Rivista di matematica elementare
Zeitschrift zur Pflege der Mathematik und zur Förderung des mathematisch-physikalischen Unterrichts
Organ für den Verein Schweizerischer Mathematiklehrer
El. Math. Band II Nr. 4 Seiten 73–88 Basel, 15. Juli 1947

Un peu de mathématiques à propos d'une courbe plane[1])

1. Les points qui divisent en trois parties égales les côtés d'un polygone fermé à n côtés P sont les sommets d'un nouveau polygone fermé P' à $2n$ côtés. Pour abréger, disons que P' se déduit de P par *trisection* (fig. 1).

Fig. 1. Trisection d'un polygone

En partant d'un carré P_0 et en répétant cette opération, on obtient une suite de polygones P_n ($n = 0, 1, \ldots$), dont chacun se déduit du précédent par trisection. Ces polygones P_n tendent vers une *courbe limite C*, convexe, qui limite la région du plan formée des points qui appartiennent à tous les polygones P_n.

Ce procédé de trisection est utilisé pour tailler les manches de marteaux et donner au profil primitivement rectangulaire de l'ébauche une forme «arrondie». Après deux ou trois opérations, un coup de lime tient lieu de passage à la limite. M. ANDRÉ AMMANN, étudiant à Genève, à qui un élève de l'Ecole des Arts et Métiers qui construisait ainsi des marteaux avait demandé «l'équation» de la courbe C, en remarqua l'intérêt, et c'est à lui que je dois le sujet de cette conférence.

Dans les n[os] 2, 3 et 4, j'établis quelques propriétés de cette courbe et je prouve qu'*elle n'a aucun arc analytique*. Cette étude fait intervenir des suites de nombres rationnels, analogues aux suites de FAREY, considérées notamment par BROCOT, HALPHEN et LUCAS, que ce dernier appelle les *suites de* BROCOT. Pour ne pas renvoyer le lecteur à d'autres ouvrages, j'établis leurs propriétés essentielles au n° 5.

Aux n[os] 6 et 7, je détermine des procédés permettant de calculer les coordonnées d'un point de C et le coefficient angulaire de la tangente, soit en fonction d'un para-

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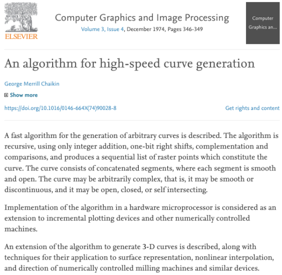

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Computer Graphics and Image Processing
Volume 3, Issue 4, December 1974, Pages 346-349

Computer Graphics an...

An algorithm for high-speed curve generation

George Merrill Chaikin

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A fast algorithm for the generation of arbitrary curves is described. The algorithm is recursive, using only integer addition, one-bit right shifts, complementation and comparisons, and produces a sequential list of raster points which constitute the curve. The curve consists of concatenated segments, where each segment is smooth and open. The curve may be arbitrarily complex, that is, it may be smooth or discontinuous, and it may be open, closed, or self intersecting.

Implementation of the algorithm in a hardware microprocessor is considered as an extension to incremental plotting devices and other numerically controlled machines.

An extension of the algorithm to generate 3-D curves is described, along with techniques for their application to surface representation, nonlinear interpolation, and direction of numerically controlled milling machines and similar devices.

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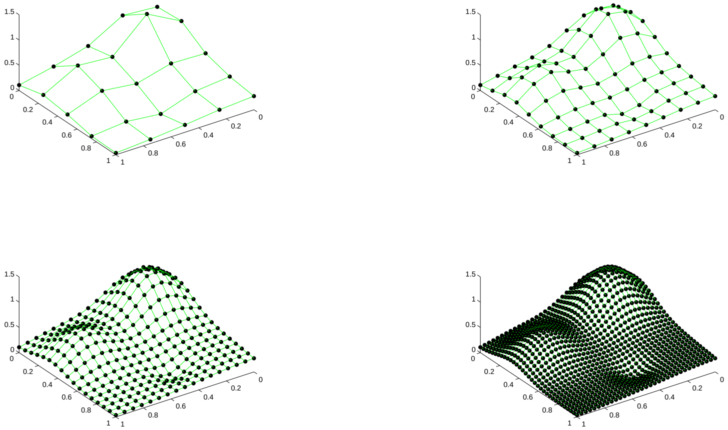
- $\lim_{k\to\infty} \{\mathcal{D}_k\}$ is the subdivision limit generated by the scheme

Subdivision schemes: the general idea

$\mathcal{D}_0$: originally a set of points but, due to their simplicity, in the last 30 years are extended to more abstract settings, such as

- polygonal meshes,
- vector fields,
- manifold valued data,
- matrices,
- sets,
- curves, nets of functions,
- .....

An example of point subdivision scheme



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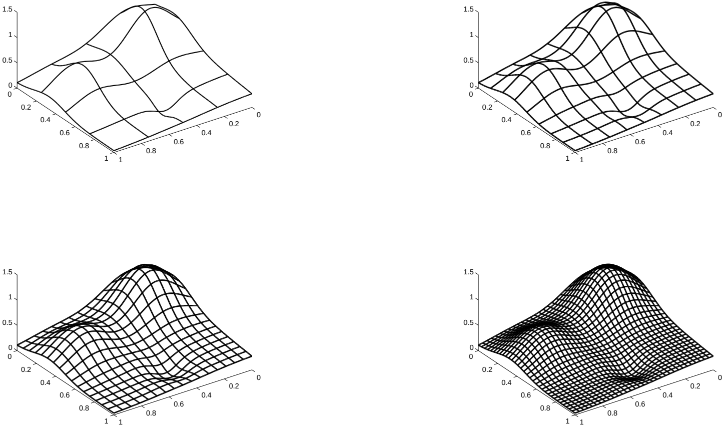
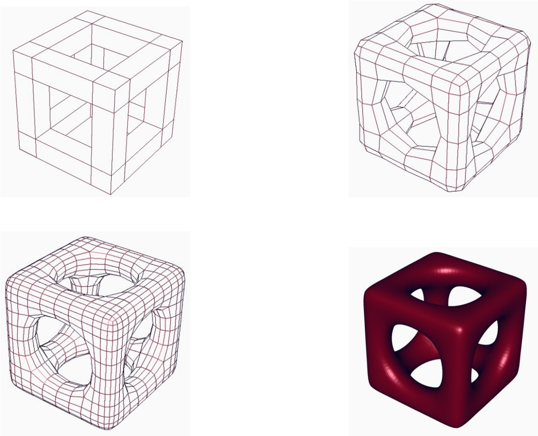


An example of mesh subdivision scheme



Why subdivision schemes?

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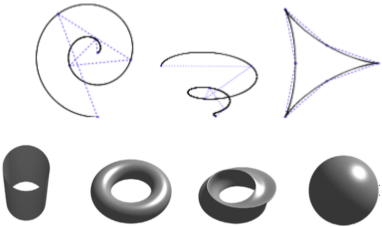
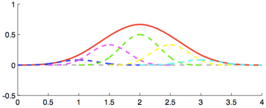
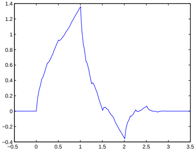
Generation of curves and surfaces: geometric modelling and CAGD



Why subdivision schemes?

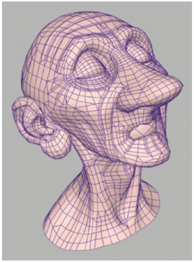
Generation of refinable functions: multiresolution analysis and wavelets





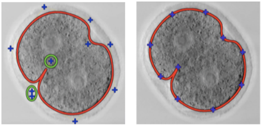
Why Subdivision Schemes?

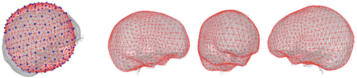
Computer animation: simple and efficient surface representation



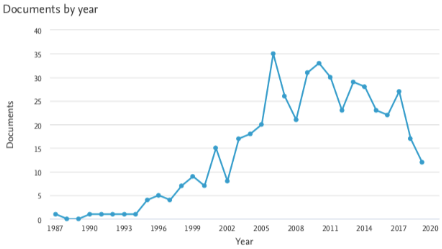
Why subdivision schemes?

Image analysis: generation of active contours and active surfaces





Subdivision nowaday: abundance of schemes



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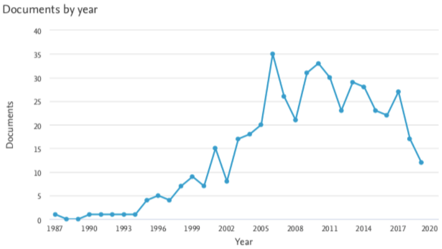

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- simplicity of definition and implementation
- computational efficiency

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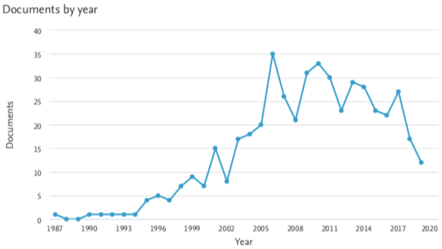

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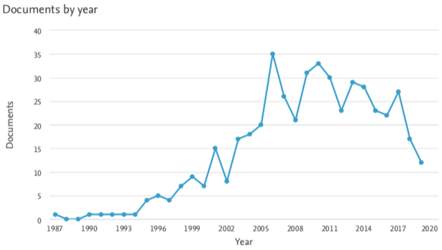

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☞ New trends: level dependent, non-linear, application driven ..... they requires news ideas and new tools for their theoretical analysis

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☞ Linear subdivision schemes acquire a strategic significance for the integration and cooperation between different areas of research

Cardinal B-splines: refinement properties

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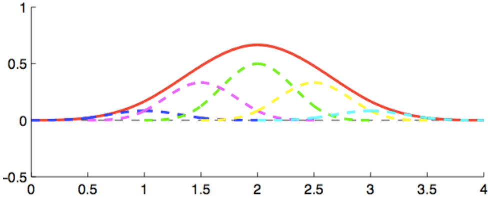

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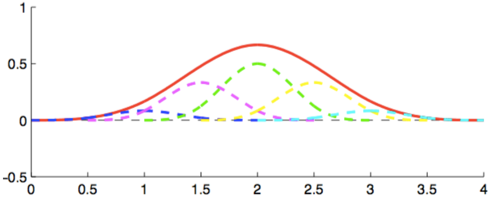

☞ Coefficients of the cubic B-spline refinement mask: $\frac{1}{8}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{8}$

The binary subdivision scheme for cardinal cubic splines

Using the refinability properties

$$B_3(t) = \sum_{j\in\mathbb{Z}} a_j^3 B_3(2t-j), \quad \text{where} \quad \mathbf{a}^3 = \cdots 0, \frac{1}{8}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{8}, 0, \cdots$$

any cubic polynomial spline can be written as

$$s(t) = \sum_{i\in\mathbb{Z}} P_i B_3(t-i) = \sum_{i\in\mathbb{Z}} P_i \sum_{j\in\mathbb{Z}} a_j^3 B_3(2(t-i)-j), \quad \text{that is}$$

$$s(t) = \sum_{i\in\mathbb{Z}} \underbrace{\left(\sum_{j\in\mathbb{Z}} a_{i-2j}^3 P_j \right)}_{P_i^{(1)}} B_3(2t-i) = \sum_{i\in\mathbb{Z}} P_i^{(1)} B_3(2t-i)$$

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where the refined set of points $P^{(k+1)},\ i \in \mathbb{Z}$ is

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☞ Since the support of $B_3(2^k\cdot)$ shrink for k large enough the coefficients $\mathbf{P}^{(k)}$ are a good discrete representation of s.

Binary subdivision scheme

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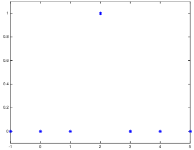

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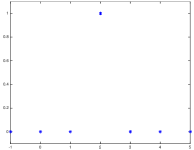

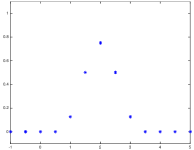

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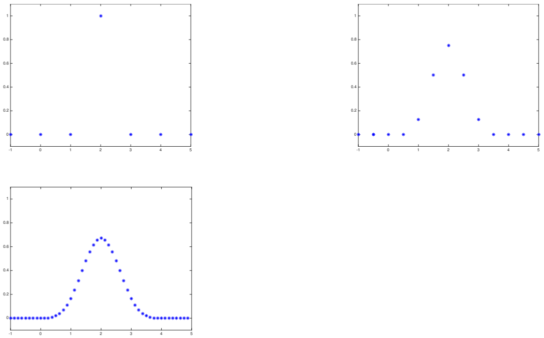

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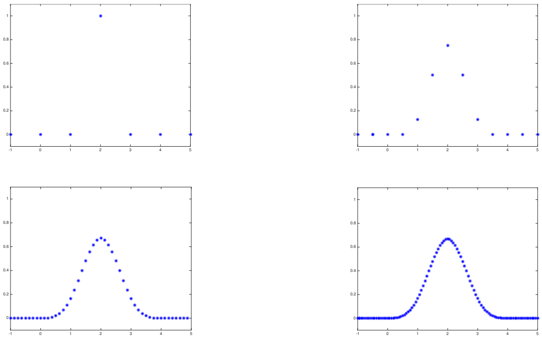

Univariate binary spline subdivision schemes: functional case

Any spline subdivision scheme iteratively computes $\mathbf{P}^{(k+1)} := S_{\mathbf{a}}\mathbf{P}^{(k)},\ k \geq 1$ via the binary subdivision operator

$$S_{\mathbf{a}} : \ell(\mathbb{Z}) \to \ell(\mathbb{Z}), \qquad \left(S_{\mathbf{a}}\mathbf{P}^{(k)}\right)_i = \sum_{j\in\mathbb{Z}} a_{i-2j}\, P_j^{(k)}, \qquad i \in \mathbb{Z},$$

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The basic subdivision operations are CONVOLUTION+ MERGE

$$\mathbf{P}_e^{(k+1)} = \mathbf{a}_e * \mathbf{P}^{(k)}, \quad \mathbf{P}_o^{(k+1)} = \mathbf{a}_o * \mathbf{P}^{(k)} \quad \to \quad \mathbf{P}^{(k+1)} = \text{merge}(\mathbf{P}_e^{(k+1)}, \mathbf{P}_o^{(k+1)})$$

Univariate binary subdivision operators: functional case

linear splines $\qquad P_{2i}^{(k+1)} = 1 P_i^{(k)}$

$\mathbf{a} = (\frac{1}{2},\ 1,\ \frac{1}{2}) \qquad P_{2i+1}^{(k+1)} = \frac{1}{2} P_i^{(k)} + \frac{1}{2} P_{i+1}^{(k)}$

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Univariate binary subdivision operators: functional case

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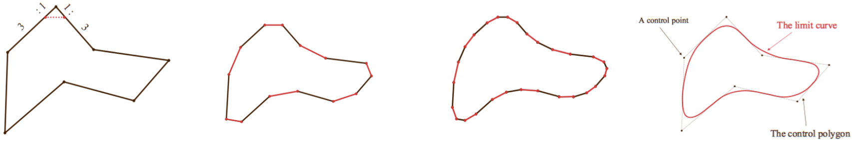

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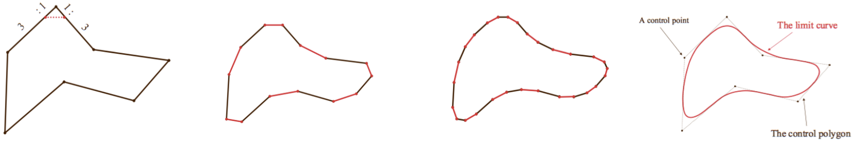

☞ From a practical point of view 6 or 7 iterations are enough to get a very good approximation of the subdivision limit (up to pixel accuracy).

Interpolatory subdivision schemes

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Interpolatory subdivision schemes

Four point scheme $\mathbf{a} = \left(\cdots, 0, -\frac{1}{16}, 0, \frac{9}{16}, 1, \frac{9}{16}, 0, -\frac{1}{16}, 0, \cdots\right)$

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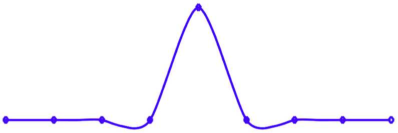

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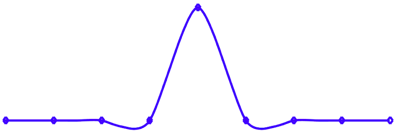
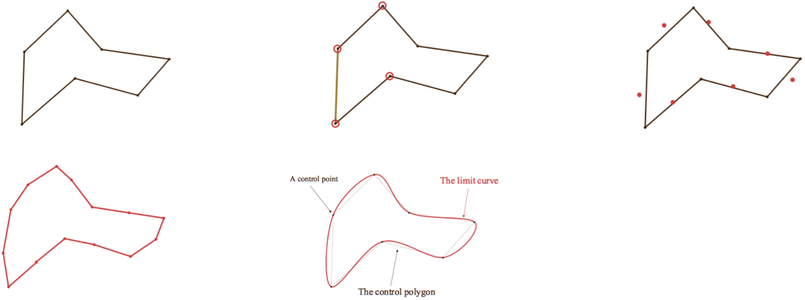


N. Dyn, D. Levin, J. A. Gregory, A 4-point interpolatory subdivision scheme for curve design, Comput. Aided Geom. Design 4 (4) (1987) 257–268.

The four point schemes



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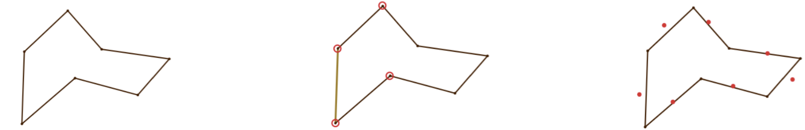

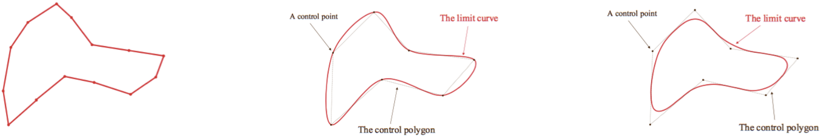

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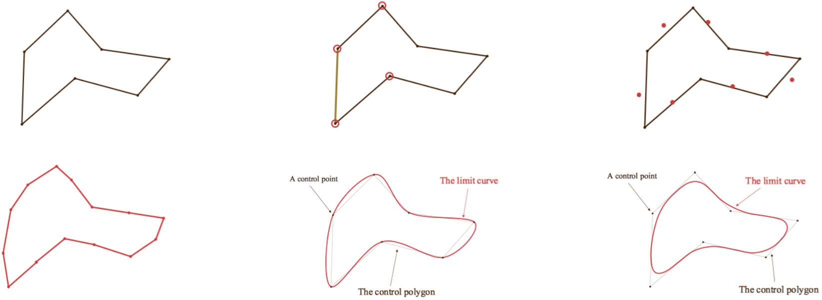

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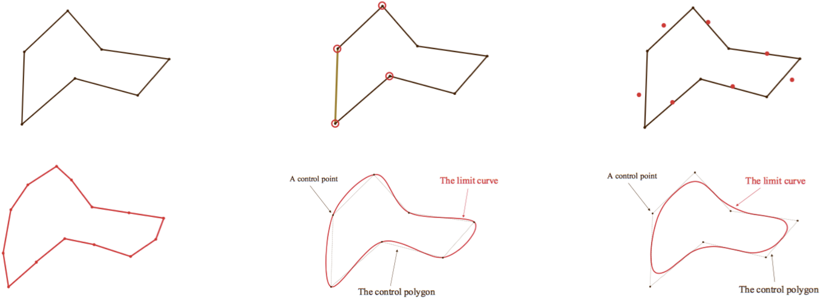

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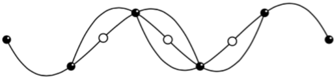

Binary linear subdivision scheme

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- Mostly, this function is *not defined analytically*. But:
  - ▷ $\phi = \sum_{i\in\mathbb{Z}} a_i \phi(2\cdot - i)$;
  - ▷ partition of unity $\sum_{i\in\mathbb{Z}} \phi(\cdot - i) = 1$;
  - ▷ has a known regularity and specific reproduction/approximation properties.

Non-stationary or level dependent subdivision scheme

Subdivision with a different set of coefficients at each level:

$$\{\mathbf{a}^{(k)},\ S_{\mathbf{a}^{(k)}},\ k \geq 0\} \quad \Leftrightarrow \quad \begin{cases} \text{Input} \quad \mathbf{P}^{(0)}, \quad \{\mathbf{a}^{(k)},\ k \geq 0\} \\ \text{For} \quad k = 0, 1, \cdots \\ \qquad \mathbf{P}^{(k+1)} := S_{\mathbf{a}^{(k)}} \mathbf{f}^{(k)} \quad \text{level dep. rules} \end{cases}$$

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☞ They still satisfy a "refinability" property: $\phi_m = \sum_{i\in\mathbb{Z}} a_i^{(m)} \phi_{m+1}(2\cdot -i).$

Rvachev-type functions

Rvachev-type functions: compactly supported C^{∞}-function

$$\begin{cases} \text{linear splines} & \mathbf{a} = (0,\ \frac{1}{2},\ 1,\ \frac{1}{2}, 0) \\ \text{quadr. splines} & \mathbf{a} = (0,\ \frac{1}{4},\ \frac{3}{4},\ \frac{3}{4}, \frac{1}{4},\ 0) \\ \text{cubic splines} & \mathbf{a} = (0,\ \frac{1}{8},\ \frac{4}{8},\ \frac{6}{8},\ \frac{4}{8},\ \frac{1}{8},\ 0) \\ \text{quartic splines} & \mathbf{a} = (0,\ \frac{1}{16},\ \frac{5}{16},\ \frac{5}{16},\ \frac{5}{16},\ \frac{5}{16},\ 0) \\ \text{quintic splines} & \mathbf{a} = (0,\ \frac{1}{32},\ \frac{6}{32},\ \frac{15}{32},\ \frac{20}{32},\ \frac{15}{32},\ \frac{6}{32},\ \frac{1}{32},\ 0) \end{cases}$$

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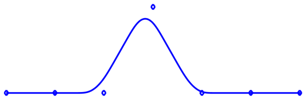
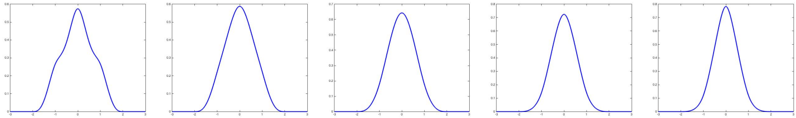


Cardinal exponential splines with segments in $\{e^{\theta t}, e^{-\theta t}, te^{\theta t}, te^{-\theta t}\}$

$$P_{2i}^{(k+1)} = \frac{1}{2(v^{(k)}+1)^2} P_{i-1}^{(k)} + \frac{4(v^{(k)})^2+2}{2(v^{(k)}+1)^2} P_i^{(k)} + \frac{1}{2(v^{(k)}+1)^2} P_{i+1}^{(k)}$$

$$P_{2i+1}^{(k+1)} = \frac{2v^{(k)}}{(v^{(k)}+1)^2} P_i^{(k)} + \frac{2v^{(k)}}{(v^{(k)}+1)^2} P_{i+1}^{(k)}$$

For $\quad v^{(k)} = \frac{1}{2}\left(e^{\frac{\theta}{2^{k+1}}} + e^{\frac{-\theta}{2^{k+1}}}\right), \quad v^{(k)} = \sqrt{\frac{1+v^{(k-1)}}{2}}, \quad k \geq 0, \quad v^{(-1)} > -1$



Basic limit functions for different values of $v^{(-1)} \in \{-0.9,\ -0.5,\ 0.5,\ 0.25,\ 0.45\}$

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Convergence (Definition)

The scheme $\{S_{\mathbf{a}^{(k)}},\ k \geq 0\}$ applied to the initial data $\mathbf{P}^{(0)} \in \ell(\mathbb{Z})$ is called *convergent* if there exists a function $f_{\mathbf{P}^{(0)}} \in C(\mathbb{R})$, $(f_{\mathbf{P}^{(0)}} \neq 0,\ \mathbf{P}^{(0)} \neq 0)$ such that

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Regularity (Definition)

The scheme $\{S_{\mathbf{a}^{(k)}},\ k \geq 0\}$ is C^ℓ−*convergent* if $f_{\mathbf{P}^{(0)}} \in C^\ell(\mathbb{R})$.

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☞ Many of the properties of a subdivision scheme can be easily checked using algebraic conditions on the subdivision symbols;This is also true for the properties of the basic limit functions including their approximation order.

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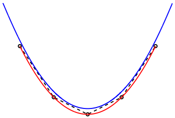
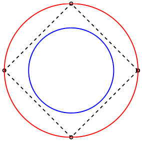
- A convergent subdivision scheme reproduces $\mathcal{V}$ if for any $f \in \mathcal{V}$ and $\mathbf{P}^{(0)} = \{f(t_i^{(0)}),\ i \in \mathbb{Z}\}$ we have $\lim_{k\to\infty} S_{\mathbf{a}^{(k)}} S_{\mathbf{a}^{(k-1)}} \cdots S_{\mathbf{a}^{(0)}} \mathbf{P}^{(0)} = f$.





Exponential polynomials: important in applications

Exponential-polynomials (Definition)

Let $n \in \mathbb{N}$ and let $\Gamma = \{(\theta_1, \xi_1), \ldots, (\theta_n, \xi_n)\}$ with $\theta_i \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$, $\theta_i \neq \theta_j$ if $i \neq j$ and $\xi_i \in \mathbb{N}, \; i = 1, \cdots, n$. We define the space of exponential polynomials EP_Γ

$$EP_\Gamma = \mathrm{span}\{ \, x^{r_i} \, e^{\theta_i x}, \; r_i = 0, \cdots, \xi_i - 1, \; i = 1, \cdots, n\} \, .$$

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For $\Gamma = \{(0, 2),\ (\theta, 2),\ (-\theta, 2)\}$ we have $EP_\Gamma = \mathrm{span}\{1,\ x,\ e^{\theta x},\ e^{-\theta x},\ xe^{\theta x},\ xe^{-\theta x}\}$.

$\theta = 0 \quad \to \mathrm{span}\{1,\ x,\ x^2,\ x^3,\ x^4,\ x^5\};$

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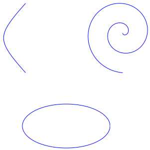

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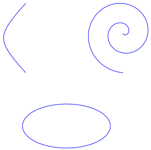

☞ Exponential polynomials cannot be reproduced by stationary subdivisions!

Generation/reproduction of a non-stationary subdivisions

Theorem [C.C. and L. Romani 2011]

Let $\Gamma = \{(\theta_1, \xi_1), \cdots, (\theta_n, \xi_n)\}$, $z_\ell^{(k)} = e^{\frac{-\theta_\ell}{2^{k+1}}}$, $\ell = 1, \cdots, n$, $k \geq 0$. A stable subdivision scheme $\{S_{\mathbf{a}^{(k)}},\ k \geq 0\}$

- generates EP_Γ iff $\frac{d^r\, a^{(k)}(-z_\ell^{(k)})}{dz^r} = 0,\ r = 0, ..., \xi_\ell - 1 \qquad (*)$
- reproduces EP_Γ with respect to $t_i^{(k)} = \frac{i+\tau}{2^k}$, iff beside $(*)$, $\frac{d^r\, a^{(k)}(z_\ell^{(k)})}{dz^r} = 2\left(z_\ell^{(k)}\right)^{\tau - r} \Pi_{q=0}^{r-1}(\tau - q), \quad r = 1, ..., \xi_\ell - 1.$

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Let $\Gamma = \{(\theta_1, \xi_1), \cdots, (\theta_n, \xi_n)\}$, $z_\ell^{(k)} = e^{\frac{-\theta_\ell}{2^{k+1}}}$, $\ell = 1, \cdots, n$, $k \geq 0$. A stable subdivision scheme $\{S_{\mathbf{a}^{(k)}},\ k \geq 0\}$

- generates EP_Γ iff $\frac{d^r\, a^{(k)}(-z_\ell^{(k)})}{dz^r} = 0,\ r = 0, ..., \xi_\ell - 1 \qquad (*)$
- reproduces EP_Γ with respect to $t_i^{(k)} = \frac{i+\tau}{2^k}$, iff beside $(*)$,
  $\frac{d^r\, a^{(k)}(z_\ell^{(k)})}{dz^r} = 2\left(z_\ell^{(k)}\right)^{\tau - r} \Pi_{q=0}^{r-1}(\tau - q), \quad r = 1, ..., \xi_\ell - 1.$

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C. Conti, L. Gemignani, L. Romani, Exponential Pseudo-Splines: looking beyond Exponential B-splines, Journal of Mathematical Analysis and Applications 439 (2016), Pages 32-56.

Subdivision for surfaces

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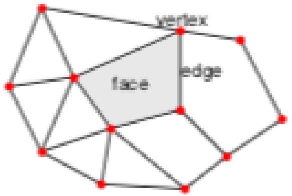
In the surface case a subdivision scheme deals with meshes $\mathcal{M} = (V, E, F)$

- V $\rightarrow$ vertices
- E $\rightarrow$ edges (pairs of vertices)
- F $\rightarrow$ faces (cyclic lists of edges)



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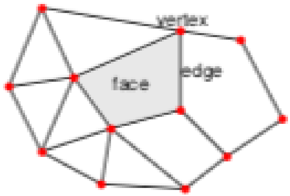

☞ A subdivision step is a mesh refinement step $\mathcal{M}^{(k)} \rightarrow \mathcal{M}^{(k+1)}$

Subdivision for surfaces: important meshes

Quadrilater mesh $\rightarrow$ all faces consist of 4 edges and 4 vertices

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regular quad-mesh:

- it is topologically equivalent to $\mathbb{Z}^2$ with edges parallel to the directions e_1, e_2
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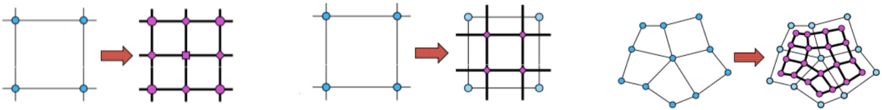
irregular quad-mesh:

- has extraordinary vertices where not 4, but 3, 5, or even more faces meet
- special rules and special analysis tools are needed near extraordinary vertices
- non-quadrilateral faces after one refinement step



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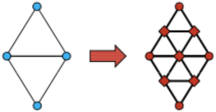

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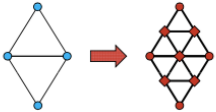


☞ a triangular mesh with some irregular vertices can describe arbitrary topology

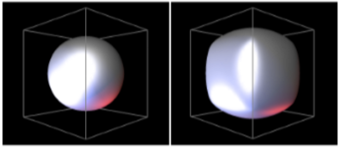
Most popular subdivision schemes for surfaces (stationary)

quadrilater meshes

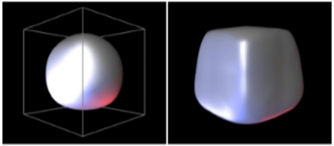
- Doo-Sabin $\rightarrow C^1$-surfaces
- Catmull-Clark $\rightarrow C^1$-surfaces (C^2 in regular regions)

triangular meshes

- Loop $\rightarrow C^1$-surfaces (C^2 in regular regions)
- Butterfly $\rightarrow C^1$-surfaces (interpolatory)



Catmull-Clark Doo-Sabin



Loop Butterfly

Analysis tools for stationary surface subdivision

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☞ Non-stationary schemes are very important and very useful!

Non-stationary subdivision scheme for surfaces

- non-stationary variant of subdivision schemes for surfaces on regular meshes can be analysed with known tools

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M. Charina, C. Conti, N. Guglielmi, V. Protasov, Regularity of Non-Stationary Subdivision: a Matrix Approach, Numer. Math. (2017) 135, Pages 639–678

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☞ With Maria Charina and Nira Dyn we are working on high-regular bivariate counterparts of the Rvachev-type UP-function based on three-directions box splines.

Non-stationary subdivision scheme for surfaces

- regular meshes are too rigid: vertices of valence different 6 and 4 are important to model even simple shapes

(a) Initial mesh (b) Limit surface of the BLOB scheme (c) Limit surface of Loop's scheme

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- there exists non-stationary variants of subdivision scheme for surfaces on irregular meshes with "gaps" in their analysis

Analysis tools for non-stationary surface subdivision

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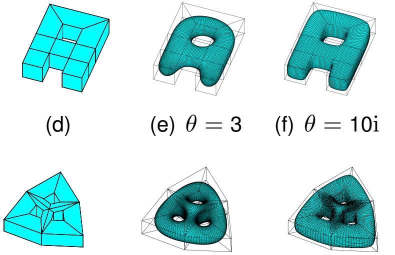
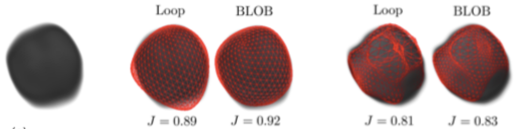


(d) (e) $\theta = 3$ (f) $\theta = 10\mathrm{i}$

(a) (b) $\theta = 3$ (c) $\theta = 10\mathrm{i}$

Non-stationary subdivision scheme for surfaces

Non-stationary schemes as the *BLOB* scheme can reproduce sphere-like structures important in many medical/biomedical applications such as delineation of organs like brain, lungs, kidneys..



(a) Real image (b) $\mathcal{M}^{(0)}$ with 6 control points (c) $\mathcal{M}^{(0)}$ with 258 control points

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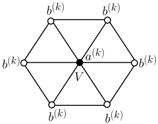
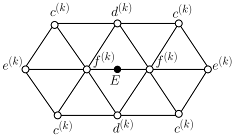
☞ BLOB: Butterfly-Loop Optimal Blending

The (level-dependent) refinement operators

1) Vertex/Edge-point rules around vertices of valence 6

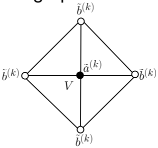


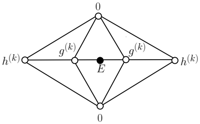


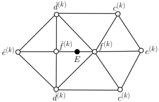
$$a^{(k)} = \frac{4(v^{(k)})^2+2v^{(k)}+1}{4(v^{(k)}+1)^2}, \qquad b^{(k)} = \frac{2v^{(k)}+1}{8(v^{(k)}+1)^2}, \qquad c^{(k)} = \frac{2v^{(k)}+1}{16(v^{(k)}+1)^3}, \quad v^{(k)} = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\frac{\lambda}{2^{(k+1)}}} + \mathrm{e}^{-\mathrm{i}\frac{\lambda}{2^{(k+1)}}}\right)$$

$$d^{(k)} = \frac{(2v^{(k)}+1)^2}{8(v^{(k)}+1)^3}, \qquad e^{(k)} = \frac{1}{16(v^{(k)}+1)^3}, \qquad f^{(k)} = \frac{(2v^{(k)}+1)(4(v^{(k)})^2+6v^{(k)}+3)}{16(v^{(k)}+1)^3},$$

2) Vertex-point and Edge-point rule around vertices of valence 4







$$\tilde{a}^{(k)} = \frac{45(v^{(k)})^2+18v^{(k)}+1}{48(v^{(k)}+1)^2}, \qquad \tilde{b}^{(k)} = \frac{3(v^{(k)})^2+78v^{(k)}+47}{192(v^{(k)}+1)^2}, \qquad g^{(k)} = \frac{(2v^{(k)}+3)(2v^{(k)}+1)}{8(v^{(k)}+1)^2}, \qquad h^{(k)} = \frac{1}{8(v^{(k)}+1)^2}$$

$$c^{(k)} = \frac{2v^{(k)}+1}{16(v^{(k)}+1)^3}, \qquad e^{(k)} = \frac{1}{16(v^{(k)}+1)^3}, \qquad f^{(k)} = \frac{(2v^{(k)}+1)(4(v^{(k)})^2+6v^{(k)}+3)}{16(v^{(k)}+1)^3},$$

$$\tilde{d}^{(k)} = \frac{16(v^{(k)})^2+18v^{(k)}+5}{32(v^{(k)}+1)^3}, \qquad \tilde{e}^{(k)} = \frac{2v^{(k)}+5}{64(v^{(k)}+1)^3}, \qquad \tilde{f}^{(k)} = \frac{32(v^{(k)})^3+64(v^{(k)})^2+54v^{(k)}+15}{64(v^{(k)}+1)^3}$$

Analysis tools for non-stationary surface subdivision

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For the analysis of subdivision schemes near extraordinary points, our idea is to extend the notion of "asymptotical equivalence" with a stationary scheme:

- Asymptotical equivalence means that two schemes differ only on the "initial" steps but, asymptotically, are the same

Asymptotical equivalence

The notion of asymptotical equivalence was only considered in the regular case

N. Dyn, D. Levin, Analysis of Asymptotically Equivalent Binary Subdivision Schemes, Journal of Mathematical Analysis and Applications,193, 2, (1995), 594-621

Definition: Asymptotical equivalence

Let $\bar{\mathcal{S}}$ and $\mathcal{S}$ be subdivision schemes based on the sub. operators $S_{\mathbf{a}}$ and $\{S_{\mathbf{a}^{(k)}},\ k \geq 1\}$ respectively. If

$$\sum_{k=1}^{+\infty} 2^{\ell k} \|S_{\mathbf{a}^{(k)}} - S_{\mathbf{a}}\|_\infty < +\infty \ \text{ with } \ \|S_{\mathbf{a}^{(k)}}\|_\infty := \max \left\{ \sum_{\beta \in \mathbb{Z}^2} |a^{(k)}_{\alpha - 2\beta}| \ : \ \beta \in \mathbb{Z}^2 \right\}$$

then $\bar{S}$ and S are said to be *asymptotically equivalent* schemes of order ℓ.

Convergence near extraordinary points/faces

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Theorem [C.C., M. Donatelli, P. Novara, L. Romani (2019)]

Assume that a non-stationary subdivision scheme $\mathcal{S} = \{S_k, k \geq 1\}$ and a stationary "reference" subdivision scheme $\bar{\mathcal{S}} = \{S\}$ satisfy:

i) $\bar{\mathcal{S}}$ is convergent both on *regular and irregular* regions;

ii) $\mathcal{S}$ is asymptotically equivalent of order 0 to $\bar{\mathcal{S}}$ on *regular* regions;

iii) on the irregular regions S_k and S satisfy, for all $k \geq 1$,

$$\|S_k - S\| \leq \frac{C}{\sigma^k} \quad \text{with} \quad \sigma > \frac{1}{\lambda_1} > 1, \quad \text{for all k } \geq 1$$

with $\lambda_1 \in \mathbb{R}_+$ is non-defective and $1 = \lambda_0 > \lambda_1 > |\lambda_i|, \quad i \geq 2$.

Then, the non-stationary subdivision scheme $\mathcal{S}$ is *convergent* also at extraordinary points/faces.

Regularity analysis for non-stationary surface subdivision

Theorem [C.C., M. Donatelli, P. Novara, L. Romani (2019)]

Assume that a non-stationary subdivision scheme $\mathcal{S} = \{S_k,\ k \geq 1\}$ and a stationary one $\bar{\mathcal{S}} = \{S\}$ are asymptotically equivalent of order 1. Assume that some further technical conditions are verified for the eigenvalues of both S and $\{S_k,\ k \geq 1\}$. If the stationary scheme $\bar{\mathcal{S}}$ is C^1-convergent on the regular part and at extraordinary points then the subdivision surface generated by S is *tangent plane continuous* at the extraordinary point.

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☞ But, the first steps of the non-stationary scheme can be used to influence the final shape of the subdivision limit in irregular regions.

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thank you
danke
謝謝
ngiyabonga
teşekkür ederim
спасибо
gracias
tapadh leat
bedankt
dank je
dziękuje
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kop khun krap
terima kasih
grazie
arigato
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merci