Abstract

Subdivision schemes are a method to generate smooth curves and surfaces by iteratively refining a mesh of points which eventually converge to a continuous limit.

For multiple subdivision schemes, in every refinement step different rules for data and mesh generation are allowed. New data is produced by convolution of old data with uniformly bounded sequences, called masks, the underlying mesh is modelled as a lattice which gets refined by expanding matrices, called dilation matrices.

We show how the joint spectral radius of a finite set of square matrices, derived from the refinement masks and dilation matrices, characterizes the convergence properties of the underlying multiple subdivision scheme.
Last we discuss the invariant polytope algorithm, and modifications thereof, which can be used to compute the joint spectral radius in reasonable time.

Multiple subdivision schemes and joint spectral radius

International Workshop on Multivariate Approximation and Geometric Modeling

Lomonosov Moscow State University

Thomas Mejstrik

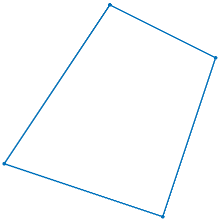
joint work with: V. Protasov, M. Charina, . . .

30th November 2021

Some examples of subdivisions

Subdivision

Chaikin's corner cutting scheme



Subdivision

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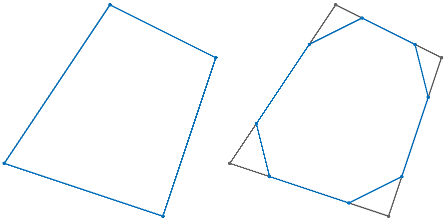

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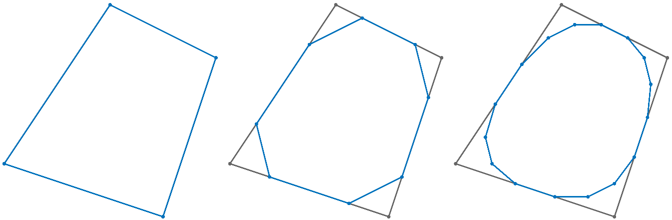

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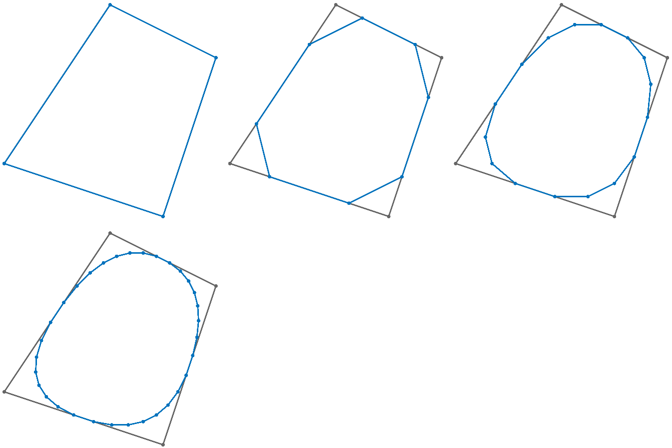

Subdivision

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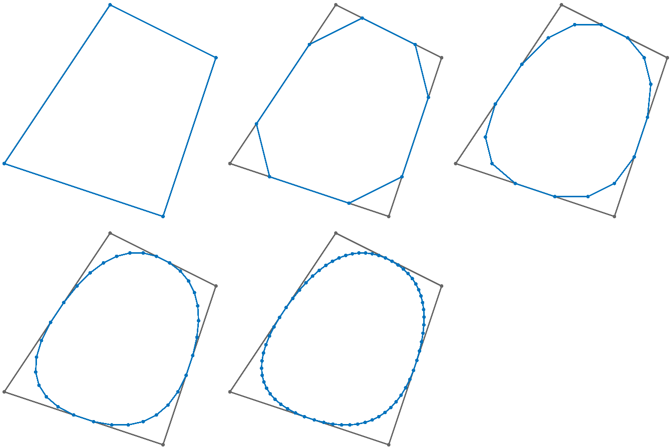

Subdivision

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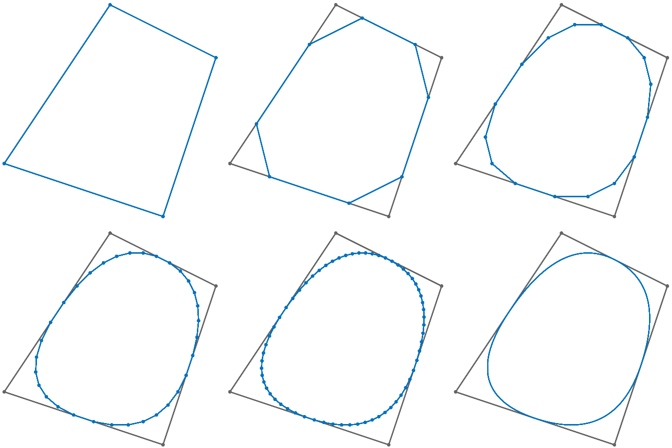


Chaikin's corner cutting scheme, coordinate wise

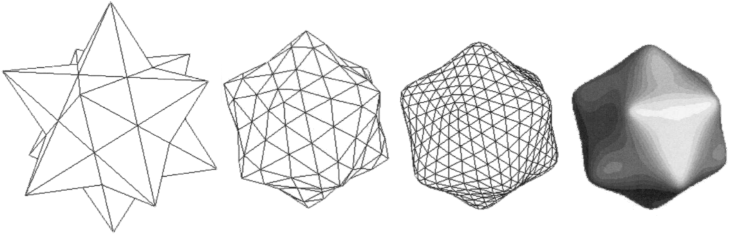
Coordinate wise.

Given

- mesh $\mathbb{Z}$
- data $c \in \ell(\mathbb{Z})$
- mesh refinement rule (*dilation matrix*) $M = \mathbb{Z} \mapsto \mathbb{Z}/2$
- data generation rules (*masks*) $a_0 = [\frac{3}{4}\ \frac{1}{4}]$, $a_1 = [\frac{1}{4}\ \frac{3}{4}]$:

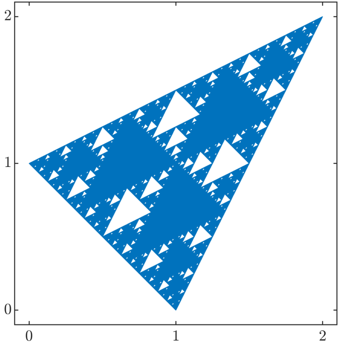
$$c \mapsto \begin{cases} c_{2\alpha} = a_0 * c \\ c_{2\alpha+1} = a_1 * c \end{cases}, \quad \alpha \in \mathbb{Z} \quad \Leftrightarrow \quad c \mapsto \sum_{\beta \in \mathbb{Z}} a(\cdot - 2\beta) c(\beta)$$

Surface subdivision: Loop scheme

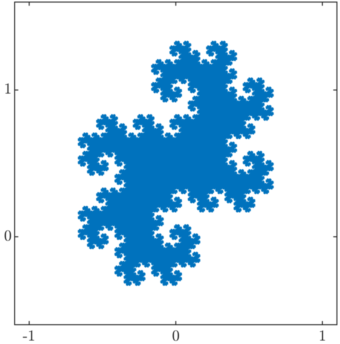


Picture: Michael Floater

Support of limit functions - Fractals

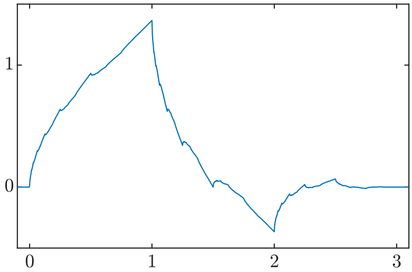


Sierpinski triangle



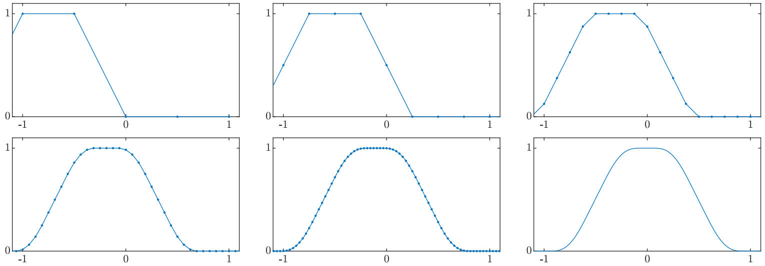
Twin dragon

Refinable functions



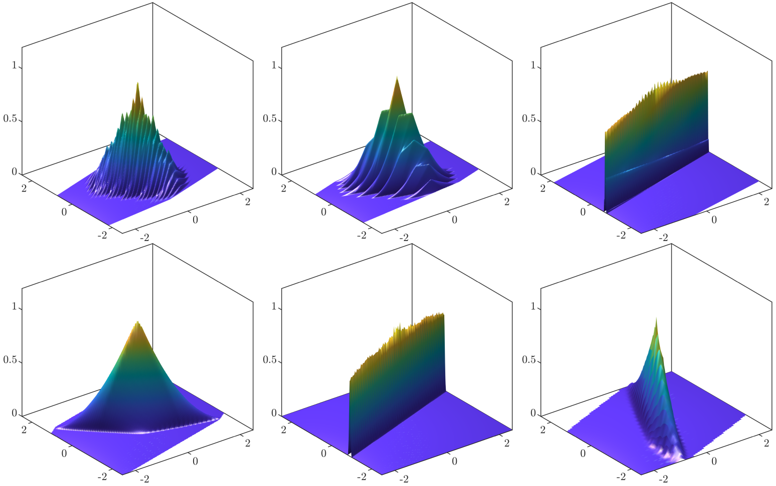
Daubechies wavelet (scaling function)

1/2 multiple schemes - variable data generation rule a



"Up-function", C^∞, finite support

Multiple schemes - variable subdivision operator



Mutually refinable functions

Leitmotif of multiple schemes

Leitmotif

Theorem [Gelfand, 41] The *spectral radius* of $A \in \mathbb{R}^{s\times s}$ is given by

$$\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

Definition [Rota, Strang, 60] The *joint spectral radius* (JSR) of $\mathcal{A} = \left\{A_j \in \mathbb{R}^{s\times s} : j = 1, \ldots, J\right\}$ is defined by

$$\mathrm{JSR}(\mathcal{A}) := \lim_{n\to\infty} \max_{(j)\in\{1,\ldots,J\}^n} \|A_{j_n} \cdots A_{j_1}\|^{1/n}.$$

- The JSR is the worst case growth rate of the norms of the matrix products.

First subject:
Multiple subdivision

Subdivision

Given $S = (a, M) \in \ell_0(\mathbb{Z}^s) \times \mathbb{Z}^{s\times s},\ \rho(M^{-1}) < 1.$

Definition The *subdivision operator* $S = (a, M)$,
is defined by $S : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)$,

$$Sc := \sum a(\cdot - M\beta)c(\beta).$$

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The sequence $S^{\mathbb{N}}$ is called a *subdivision scheme*.

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The sequence $S^{\mathbb{N}}$ is called a *subdivision scheme*.

Definition S is *convergent* if
there exist $\phi \in C_c^0(\mathbb{R}^s)$, such that

$$\lim_{n\to\infty} S^n\delta = \phi, \qquad \phi = \sum a(\alpha)\phi(M\alpha - \cdot)$$

Multiple subdivision

Given $\mathcal{S} = \{(a_j, M_j) \in \ell_0(\mathbb{Z}^s) \times \mathbb{Z}^{s\times s} : j = 1, \ldots, J\}$, $\mathrm{JSR}(\{M_j{}^{-1}\}) < 1$.

Definition The *subdivision operator* $S_j = (a_j, M_j)$, $j \in \{1, \ldots, J\}$, is defined by $S_j : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)$,

$$S_j c := \sum a_j(\cdot - M_j \beta) c(\beta).$$

A sequence $(S_j) \in \mathcal{S}^{\mathbb{N}}$ is called a *subdivision scheme*.

Definition $\mathcal{S}$ is *convergent* if for every index-sequence $(j) \in \{1, \ldots, J\}^{\mathbb{N}}$ there exists $\phi^{(j)} \in C_c^0(\mathbb{R}^s)$, such that

$$\lim_{n\to\infty} S_{j_n} \cdots S_{j_2} S_{j_1} \delta = \phi^{(j)}, \qquad \phi^{(j)} = \ ?$$

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$$\lim_{n\to\infty} S_{j_n} \cdots S_{j_2} S_{j_1} \delta = \phi_1^{(j)}, \qquad \phi_n^{(j)} = \sum a_{j_n}(\alpha) \phi_{n+1}^{(j)}(M_{j_n}\alpha - \,\cdot\,)$$

$$\lim_{n\to\infty} S_{j_n} \cdots S_{j_3} S_{j_2} \delta = \phi_2^{(j)}$$

Matrix refinement equation

Single case

Given $S = (a, M) \in \ell_0(\mathbb{Z}^s) \times \mathbb{Z}^{s\times s}$, $\rho(M^{-1}) < 1$, and define a *digit set* $D \simeq \mathbb{Z}^s / M\mathbb{Z}^s$.

Let $y = \omega + x$, with $\omega \in \mathbb{Z}^s$, $x \in X$, where

$$x = M^{-1}d_1 + M^{-2}d_2 + \cdots = 0.d_1 d_2 \ldots, \qquad d_n \in D,$$
$$X = \mathrm{cl}(M^{-1}D + M^{-2}D + \cdots).$$

$$\phi(y) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(My - \alpha).$$

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$M = 10$, $D = \{0, \ldots, 9\}$, $X = [0, 1]$

Matrix refinement equation

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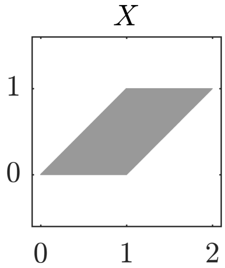
$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \; D = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$



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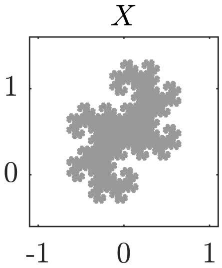

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$$\begin{aligned} x &= M^{-1} d_1 + M^{-2} d_2 + \cdots = 0.d_1 d_2 \ldots, \quad d_n \in D, \\ X &= \mathrm{cl}(M^{-1} D + M^{-2} D + \cdots). \end{aligned}$$

Then,

$$\phi(\omega + x) = \sum_{\alpha \in \mathbb{Z}^s} a(M\omega - \alpha + d_1)\phi(Mx - d_1 + \alpha)$$

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$$= T_{d_1}\Big[\phi(Mx - d_1 + \omega)\Big]_{\omega\in\mathbb{Z}^s}$$

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$$= T_{d_1} T_{d_2}\Big[\phi(.d_3 d_4 \ldots + \omega)\Big]_{\omega\in\mathbb{Z}^s}$$

Matrix refinement equation

Multiple case

Given $\mathcal{S} = \{(a_j, M_j) \in \ell_0(\mathbb{Z}^s) \times \mathbb{Z}^{s\times s} : j = 1, \dots, J\}$, $\mathrm{JSR}(\{M_j^{-1}\}) < 1$, and define *digit sets* $D_j \simeq \mathbb{Z}^s / M_j \mathbb{Z}^s$, $j = 1, \dots, J$.

Let $y = \omega + x$, with $\omega \in \mathbb{Z}^s$, $x \in X$ and $(j) \in \{1, \dots, J\}^{\mathbb{N}}$, where

$$x = M_{j_1}^{-1} d_1 + M_{j_1}^{-1} M_{j_2}^{-1} d_2 + \cdots = 0.d_1 d_2 \dots, \qquad d_n \in D_{j_n},$$
$$X^{(j)} = \mathrm{cl}(M_{j_1}^{-1} D_{j_1} + M_{j_1}^{-1} M_{j_2}^{-1} D_{j_2} + \cdots).$$

Then,

$$\left[\phi_n^{(j)}(\omega + x)\right]_{\omega\in\mathbb{Z}^s} = \Big[\sum_{\alpha\in\mathbb{Z}^s} a_{j_n}(M_{j_n}\omega - \alpha + d_1)\phi_{n+1}^{(j)}(M_{j_n}x - d_1 + \alpha)\Big]_{\omega\in\mathbb{Z}^s}$$
$$= T_{d_1, j_n}\left[\phi_{n+1}^{(j)}(.d_2 d_3 \dots + \omega)\right]_{\omega\in\mathbb{Z}^s}$$
$$= T_{d_1, j_n} T_{d_2, j_n}\left[\phi_{n+2}^{(j)}(.d_3 d_4 \dots + \omega)\right]_{\omega\in\mathbb{Z}^s}$$

Characterization of convergence: Main Result

Multiple case

Given $\mathcal{S} = \{(a_j, M_j) \in \ell_0(\mathbb{Z}^s) \times \mathbb{Z}^{s\times s} : j = 1, \ldots, J\}$,
$\mathrm{JSR}(\{M_j^{-1}\}) < 1$, and digit sets $D_j \simeq \mathbb{Z}^s / M_j \mathbb{Z}^s$, $j = 1, \ldots, J$.

Theorem [M.] There exists $\Omega \subseteq \mathbb{Z}^s$, finite, such that

$$T_{d,j} : \ell_\Omega(\mathbb{Z}^s) \to \ell_\Omega(\mathbb{Z}^s) \text{ for all } d \in D_j,\ j = 1, \ldots, J.$$

- Moreover, $\operatorname{supp} \phi_n^{(j)} \subseteq \Omega + X$ for all $n \in \mathbb{N}$, $(j) \in \{1, \ldots, J\}^{\mathbb{N}}$.
- With $V = \mathbf{1}_\Omega^\perp$,

$$\mathcal{S} \text{ is convergent} \Leftrightarrow \mathrm{JSR}(\{ T_{d,j}|_V \}) < 1,$$

Proof $\mathrm{RSR}(\mathcal{S}|_\nabla) = \mathrm{JSR}(\{ T_{d,j}|_V \})$.

Technicalities

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$$\mathrm{JSR}(\{T_{d,j}|_V\}) = \lim_{n\to\infty} \max_{d_i \in D_{j_i}, j_i} \| T_{d_n,j_n} \cdots T_{d_1,j_1}|_V \|^{1/n}$$

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$$\lim_{n\to\infty} \max_{(j)\in\{1,\dots,J\}^n} \max_{1\leqslant r\leqslant s} \sup_{\substack{c\in\ell_\infty(\mathbb{Z}^s) \\ \|\nabla_r c\|_\infty=1}} \|\nabla_r S_{j_n} \cdots S_{j_1} c\|_\infty^{1/n} = \mathrm{RSR}(\mathcal{S}|_\nabla).$$

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$$V = \mathrm{span}\{\nabla_r \delta(\cdot - \beta) \in \ell_\Omega(\mathbb{Z}^s) : 1 \leqslant r \leqslant s,\, \beta \in \mathbb{Z}^s\}.$$

$$\lim_{n\to\infty} \max_{(j)\in\{1,\ldots,J\}^n} \max_{1\leqslant r\leqslant s} \sup_{\substack{c\in\ell_\infty(\mathbb{Z}^s)\\ \|\nabla_r c\|_\infty=1}} \|\nabla_r S_{j_n} \cdots S_{j_1} c\|_\infty^{1/n} = \mathrm{RSR}(\mathcal{S}|_\nabla).$$

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$$\mathrm{JSR}(\{T_{d,j}|_V\}) = \lim_{n\to\infty} \max_{d_i \in D_{j_i},j_i} \| T_{d_n,j_n} \cdots T_{d_1,j_1}|_V\|^{1/n}, \qquad V = \mathbf{1}_\Omega^\perp.$$

$$V = \mathrm{span}\{\nabla_r \delta(\cdot - \beta) \in \ell_\Omega(\mathbb{Z}^s) : 1 \leqslant r \leqslant s,\, \beta \in \mathbb{Z}^s\}.$$

$$\tfrac{2C(\Omega)}{s(1+s)} \| T_{d,j}|_V\|_\infty \leqslant \max_{1 \leqslant r \leqslant s} \|\nabla_r S_j \delta\|_\infty \leqslant 2\, \| T_{d,j}|_V\|_\infty, \qquad \forall d,\ j.$$

$$\lim_{n\to\infty} \max_{(j)\in\{1,\dots,J\}^n} \max_{1\leqslant r\leqslant s} \sup_{\substack{c \in \ell_\infty(\mathbb{Z}^s) \\ \|\nabla_r c\|_\infty = 1}} \|\nabla_r S_{j_n} \cdots S_{j_1} c\|_\infty^{1/n} = \mathrm{RSR}(\mathcal{S}|_\nabla).$$

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$$\mathrm{JSR}(\{T_{d,j}|_V\}) = \lim_{n\to\infty} \max_{d_i\in D_{j_i},j_i} \|T_{d_n,j_n}\cdots T_{d_1,j_1}|_V\|^{1/n}, \qquad V = \mathbf{1}_\Omega^\perp.$$

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$$\frac{1}{2}\|\nabla_r S_j\delta\|_\infty \leqslant \sup_{\substack{c\in\ell_\infty(\mathbb{Z}^s)\\ \|\nabla_r c\|_\infty=1}} \|\nabla_r S_j c\|_\infty \leqslant C(\mathcal{S})\|\nabla_r S_j\delta\|_\infty, \quad \forall r,\ j.$$

$$\lim_{n\to\infty} \max_{(j)\in\{1,\dots,J\}^n} \max_{1\leqslant r\leqslant s} \sup_{\substack{c\in\ell_\infty(\mathbb{Z}^s)\\ \|\nabla_r c\|_\infty=1}} \|\nabla_r S_{j_n}\cdots S_{j_1}c\|_\infty^{1/n} = \mathrm{RSR}(\mathcal{S}|_\nabla).$$

Second subject:
Computational aspects of the joint spectral radius

Joint spectral radius

Definition [Rota, Strang, 60] Let $\mathcal{A} = \left\{ A_j \in \mathbb{R}^{s \times s} : j = 1, \dots, J \right\}$. The JSR of $\mathcal{A}$ is defined by

$$\mathrm{JSR}(\mathcal{A}) := \lim_{n \to \infty} \max_{(j) \in \{1,\dots,J\}^n} \|A_{j_n} \cdots A_{j_1}\|^{1/n}.$$

- The JSR is the worst case growth rate of the norms of the matrix products.

Joint spectral radius

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- JSR $\leqslant 1$ is undecidable [Blondel, Tsitsiklis, 00].

Computation of the JSR

Exact computation

- **I**nvariant **p**olytope **a**lgorithm [Guglielmi, M., Protasov, Wirth, Zennaro, . . .]
- Infinite **tree** algorithm [Möller, Reif]

Computation of the joint spectral radius

Theorem [Daubechies, Lagarias, 92] Given $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \ldots, J\}$. If $k \in \mathbb{N}$ and $\|\cdot\|$ is sub-multiplicative, then

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• Theorem holds for any fixed k, in particular for $k = 1$, and for any norm $\|\cdot\|$

$$\mathrm{JSR}(\mathcal{A}) \leqslant \max_j \|A_j\|$$

Invariant polytope algorithm : Idea

- Given: $A = \begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 2 & -3 \end{bmatrix}$

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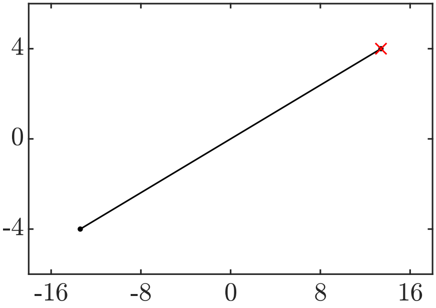

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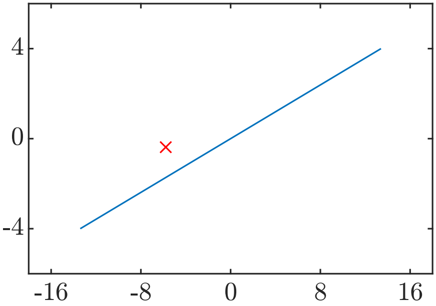

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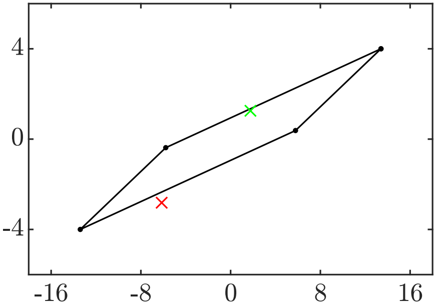

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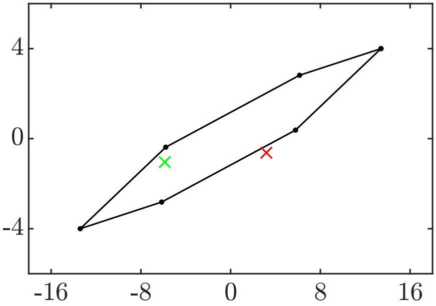

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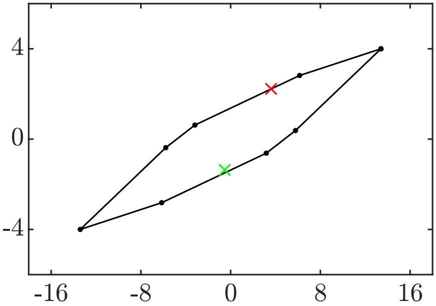

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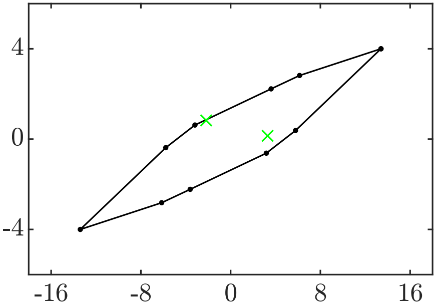


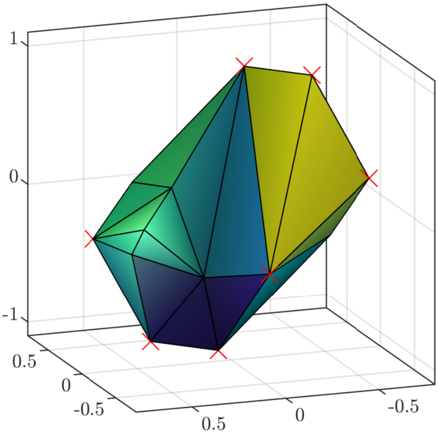
$\widetilde{A} P \subseteq P$, $\widetilde{B} P \subseteq P$,
$P = co \pm \{v_0, \widetilde{A} v_0, \widetilde{B}\widetilde{A} v_0, \widetilde{A}\widetilde{B}\widetilde{A} v_0, \widetilde{B}\widetilde{A}\widetilde{B}\widetilde{A} v_0\}$.

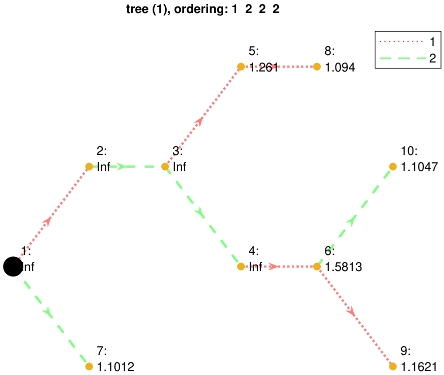
$\Rightarrow \|\widetilde{A}\|_P \leqslant 1$, $\|\widetilde{B}\|_P \leqslant 1$

$\Rightarrow \mathrm{JSR}(\{\widetilde{A}, \widetilde{B}\}) = \max\{\|\widetilde{A}\|_P, \|\widetilde{B}\|_P\} \leqslant 1$

Invariant polytope algorithm : 3D example

- Given: $A = \begin{bmatrix} 0 & 4 & 4 \\ 0 & 4 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $\quad B = \begin{bmatrix} -3 & 2 & 0 \\ -3 & 0 & 0 \\ 3 & 3 & -3 \end{bmatrix}$
- $\Pi = ABBB$





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1. There exist only finitely many products attaining the JSR (s.m.p.s), up to powers and cycles
2. The s.m.p.s have only one simple leading eigenvector (up to complex conjugates)
3. The set $\mathcal{A}$ has a spectral gap at $\mathrm{JSR}(\mathcal{A})$

Development section:
Advances

Step 1: Modified Gripenberg algorithm [M.]

$$\max_{(j)\in\{1,\ldots,J\}^k} \rho(A_{j_k}\cdots A_{j_1})^{1/k} \leqslant \mathrm{JSR}(\mathcal{A}) \leqslant \max_{(j)\in\{1,\ldots,J\}^k} \|A_{j_k}\cdots A_{j_1}\|^{1/k}.$$

• Modified Gripenberg algorithm sorts out matrix products, of which we hope that they have averaged spectral radius less than the joint spectral radius of the given set.

• Number of matrix products to be computed stays bounded w.r.t. product length.

Test results for the modified Gripenberg algorithm

$\mathcal{A} = \{A_j \in \mathbb{R}^{8\times 8} : \text{random entries}, j = 1, \ldots, 8\}$,
100 test runs

Algorithm	runtime	success rate
mod. Gripenberg	$5.4\,s$	100%
Gripenberg	$82.1\,s$	100%
genetic	$12.0\,s$	87%
brute force	$\geqslant 180.0\,s$	74%
ipa	$\gg 180.0\,s$	100%

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- Parallelisation / Warm Start of LP / Informed search
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  $\Rightarrow$ Solves most problems in low dimensions

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  $\Rightarrow$ Solves most problems in low dimensions
- Augment input with matrices of infinite powers
  $\Rightarrow$ Solves problems with multiple s.m.p.s
- Relax conditions on norm/polytope to be searched for
  $\Rightarrow \mathrm{JSR}(\mathcal{A}) \leqslant \max_{(j_i)} \|A_{j_K} \cdots A_{j_1}\|_P^{1/K}$

Development section: Applications

The Matlab t-toolboxes

```
S = getS( {1/3*[1 2 3 2 1],[1 1;1 -2]} )
```

- Download: *gitlab.com/tommsch/ttoolboxes*

The Matlab t-toolboxes

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S = getS( {1/3*[1 2 3 2 1],[1 1;1 -2]} )
blf( S ), tile( S ), tile( S, 'supp' )
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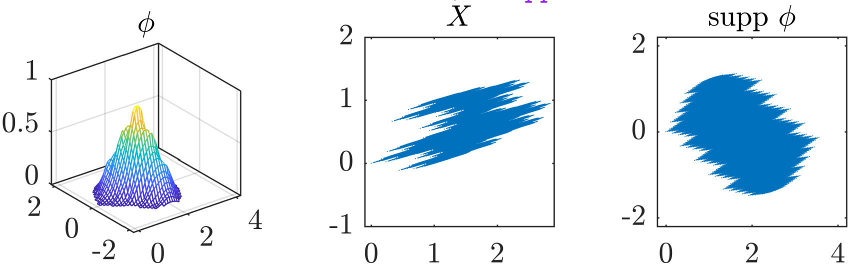

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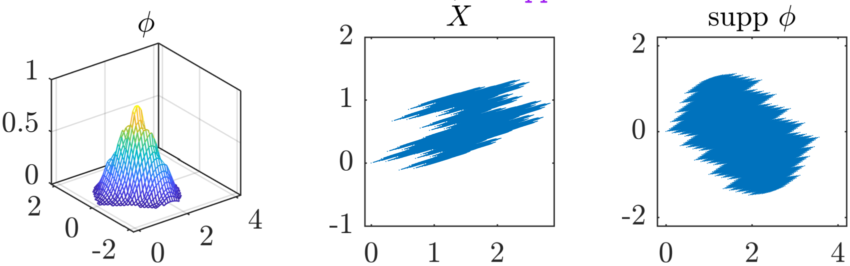


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[T, Om] = transitionmatrix( S )
V = constructVt( Om, 0 )
TV = restrictmatrix( T, V )  % 3 matrices of dim. 12
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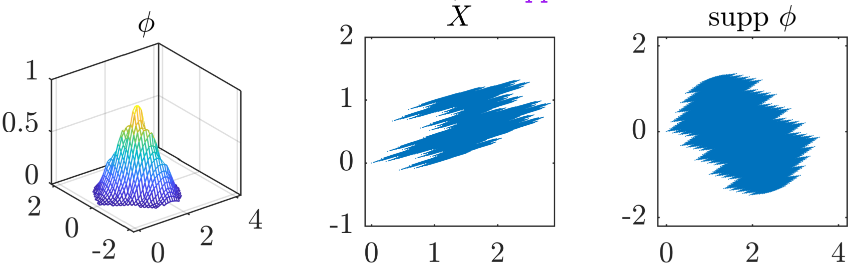


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tjsr( TV )  % (1+√13)/6 ≃ 0.7676 < 1
```

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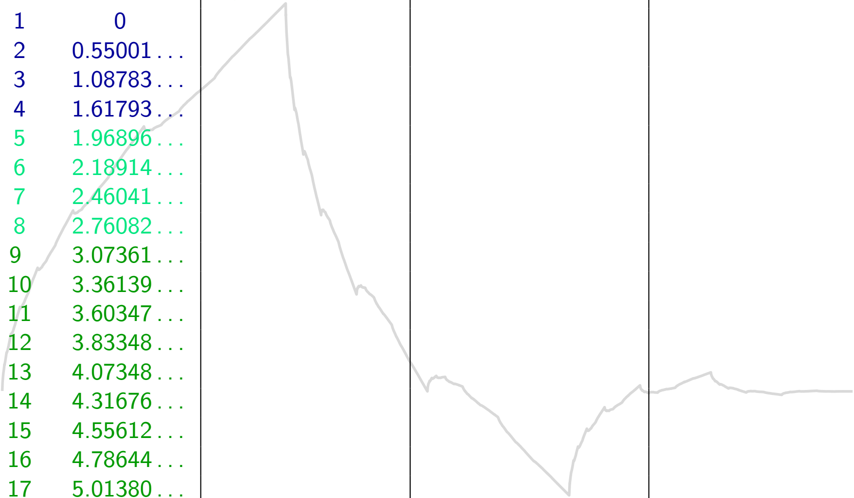
Hölder regularity α of Daubechies wavelets

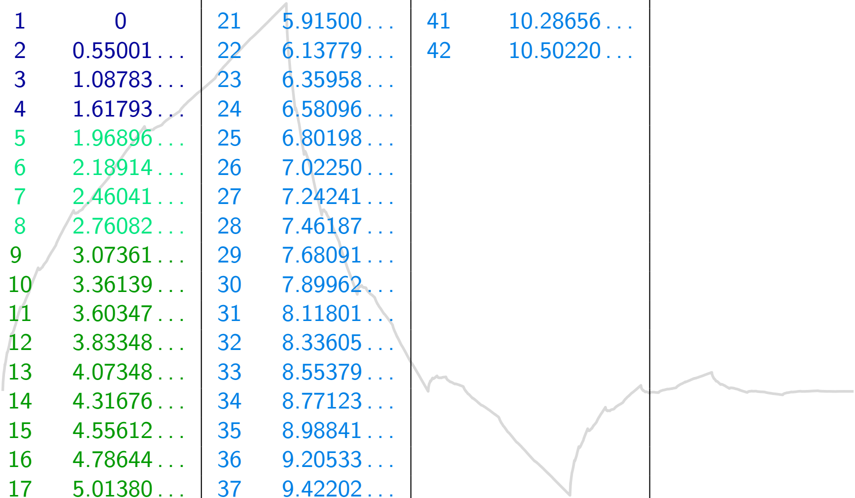
n	α
1	0
2	0.55001 ...
3	1.08783 ...
4	1.61793 ...
5	1.96896 ...
6	2.18914 ...
7	2.46041 ...
8	2.76082 ...
9	3.07361 ...
10	3.36139 ...
11	3.60347 ...
12	3.83348 ...
13	4.07348 ...
14	4.31676 ...
15	4.55612 ...
16	4.78644 ...
17	5.01380 ...
18	5.23917 ...
19	5.46532 ...
20	5.69108 ...



Daubechies, Lagarias, [92]; Gripenberg, [96]; Guglielmi, Protasov, [16];

Hölder regularity α of Daubechies wavelets

n	α	20	5.69108 . . .	40	10.07073 . . .
1	0	21	5.91500 . . .	41	10.28656 . . .
2	0.55001 . . .	22	6.13779 . . .	42	10.50220 . . .
3	1.08783 . . .	23	6.35958 . . .		
4	1.61793 . . .	24	6.58096 . . .		
5	1.96896 . . .	25	6.80198 . . .		
6	2.18914 . . .	26	7.02250 . . .		
7	2.46041 . . .	27	7.24241 . . .		
8	2.76082 . . .	28	7.46187 . . .		
9	3.07361 . . .	29	7.68091 . . .		
10	3.36139 . . .	30	7.89962 . . .		
11	3.60347 . . .	31	8.11801 . . .		
12	3.83348 . . .	32	8.33605 . . .		
13	4.07348 . . .	33	8.55379 . . .		
14	4.31676 . . .	34	8.77123 . . .		
15	4.55612 . . .	35	8.98841 . . .		
16	4.78644 . . .	36	9.20533 . . .		
17	5.01380 . . .	37	9.42202 . . .		
18	5.23917 . . .	38	9.63847 . . .		
19	5.46532 . . .	39	9.85474 . . .		



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Hölder regularity α of Daubechies wavelets

n	α	20	5.69108 . . .	40	10.07073 . . .	60	$\leqslant$ 14.35900 . . .	80
1	0	21	5.91500 . . .	41	10.28656 . . .	61	$\leqslant$ 14.57219 . . .	81
2	0.55001 . . .	22	6.13779 . . .	42	10.50220 . . .	62	$\leqslant$ 14.78531 . . .	82
3	1.08783 . . .	23	6.35958 . . .	43	$\leqslant$ 10.71766 . . .	63	$\leqslant$ 14.99833 . . .	83
4	1.61793 . . .	24	6.58096 . . .	44	$\leqslant$ 10.93295 . . .	64	$\leqslant$ 15.21127 . . .	84
5	1.96896 . . .	25	6.80198 . . .	45	$\leqslant$ 11.14807 . . .	65	$\leqslant$ 15.42413 . . .	85
6	2.18914 . . .	26	7.02250 . . .	46	$\leqslant$ 11.36304 . . .	66	$\leqslant$ 15.63692 . . .	86
7	2.46041 . . .	27	7.24241 . . .	47	$\leqslant$ 11.57785 . . .	67	$\leqslant$ 15.84962 . . .	87
8	2.76082 . . .	28	7.46187 . . .	48	$\leqslant$ 11.79252 . . .	68	$\leqslant$ 16.06226 . . .	88
9	3.07361 . . .	29	7.68091 . . .	49	$\leqslant$ 12.00705 . . .	69	$\leqslant$ 16.27482 . . .	89
10	3.36139 . . .	30	7.89962 . . .	40	$\leqslant$ 12.22144 . . .	70	$\leqslant$ 16.48731 . . .	90
11	3.60347 . . .	31	8.11801 . . .	51	$\leqslant$ 12.43571 . . .	71	$\leqslant$ 16.69973 . . .	91
12	3.83348 . . .	32	8.33605 . . .	52	$\leqslant$ 12.64985 . . .	72	$\leqslant$ 16.91209 . . .	92
13	4.07348 . . .	33	8.55379 . . .	53	$\leqslant$ 12.86387 . . .	73	$\leqslant$ 17.12438 . . .	93
14	4.31676 . . .	34	8.77123 . . .	54	$\leqslant$ 13.07778 . . .	74	$\leqslant$ 17.33661 . . .	94
15	4.55612 . . .	35	8.98841 . . .	55	$\leqslant$ 13.29157 . . .	75	$\leqslant$ 17.54878 . . .	95
16	4.78644 . . .	36	9.20533 . . .	56	$\leqslant$ 13.50526 . . .	76	$\leqslant$ 17.76089 . . .	96
17	5.01380 . . .	37	9.42202 . . .	57	$\leqslant$ 13.71884 . . .	77	$\leqslant$ 17.97295 . . .	97
18	5.23917 . . .	38	9.63847 . . .	58	$\leqslant$ 13.93232 . . .	78	$\leqslant$ 18.18494 . . .	98
19	5.46532 . . .	39	9.85474 . . .	59	$\leqslant$ 14.14571 . . .	79	$\leqslant$ 18.39688 . . .	99

Daubechies, Lagarias, [92]; Gripenberg, [96]; Guglielmi, Protasov, [16]; M.; M.;

Finiteness conjecture

Theorem [Jungers] The finiteness conjecture holds for all non-negative rational matrices, iff it holds for all pairs of $\{0,1\}$-matrices.

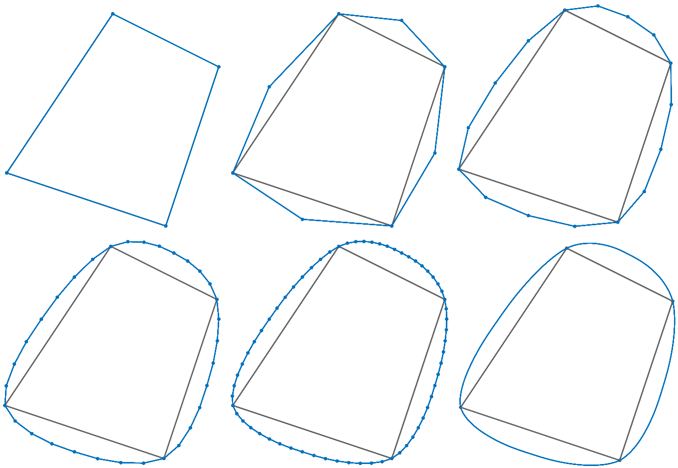
Theorem [M.] The finiteness conjecture holds for all pairs of $\{0,1\}$ matrices with dimension 3.

Thank you

More Examples of subdivision schemes

Subdivision

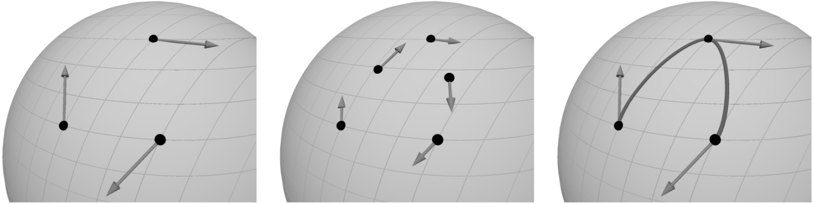
Four point scheme



Surface subdivision: Loop scheme

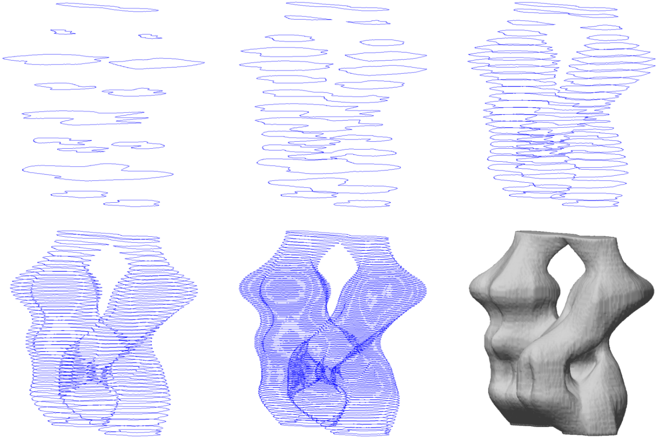
Pixar, Geri's game, 1997

Hermite subdivision on manifolds



Caroline Mossmüller, 2017

Set valued subdivision



Nira Dyn, Shay Kels, 2011

More Theory

Refinable functions

Single case

Given $S = (a, M) \in \ell_0(\mathbb{Z}^s) \times \mathbb{Z}^{s \times s}$,
$\rho(M^{-1}) < 1$.

Question Does there exist a
refinable function $\phi \in C_c^0(\mathbb{R}^s)$ such that

$$\phi(y) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \phi(My - \alpha).$$

Applications

- Processing of images
- Multigrid methods for solving PDEs

Refinable functions

Multiple case

Given $\mathcal{S} = \{(a_j, M_j) \in \ell_0(\mathbb{Z}^s) \times \mathbb{Z}^{s\times s} : j = 1, \ldots, J\}$,
$\mathrm{JSR}(\{M_j^{-1}\}) < 1$.

Question Choose any index-sequence $(j) \in \{1, \ldots, J\}^{\mathbb{N}}$. Do there exist *jointly refinable* functions $\phi_n^{(j)} \in C_c^0(\mathbb{R}^s)$, $n \in \mathbb{N}$, such that for all $n \in \mathbb{N}$

$$\phi_n^{(j)}(y) = \sum_{\alpha\in\mathbb{Z}^s} a_{j_n}(\alpha)\phi_{n+1}^{(j)}(M_{j_n} y - \alpha).$$

Applications

- Processing of images with directional features
- Multigrid methods for solving anisotropic PDEs

How to find Ω

Single case

Theorem [Cabrelli, Heil, Molter, 92]
Given $\lambda(X) = 1$. Then, $\Omega = \{\omega \in \mathbb{Z}^s : \omega + X \cap \operatorname{supp} \phi \neq \emptyset\}$.

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Example [Cotronei, Ghisi, Rossini, Sauer, 15] $S = (a, M)$,
$a = \frac{1}{3} \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$, $D = \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\} \simeq \mathbb{Z}^2 / M\mathbb{Z}^2$.

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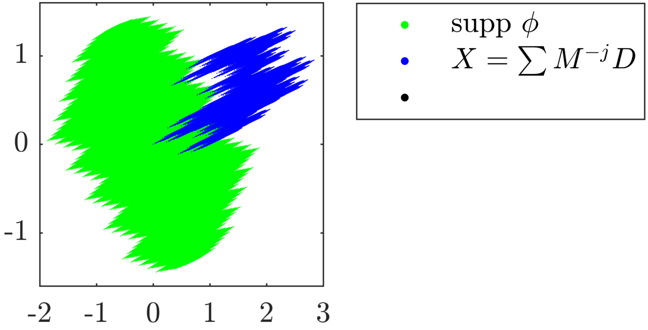

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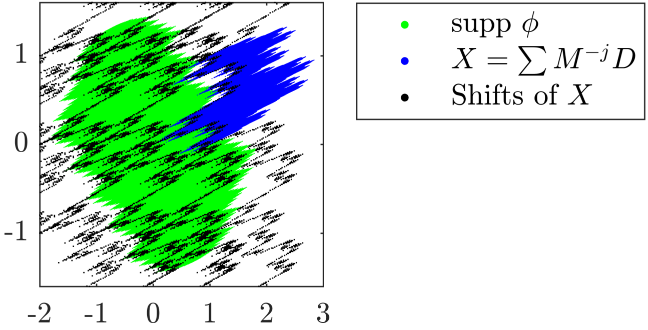

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Theorem [M.] The set Ω exists always and can be constructed iteratively. Furthermore, if $\mathcal{S}$ is convergent, then there a exists a unique smallest set Ω.