Interpolation and Approximation with Multivariate Polynomials and Splines

Ulrich Reif
Technische Universität Darmstadt

Workshop on Multivariate Approximation and Geometric Modeling

Moscow, December 01, 2021

Motivation

Modern approximation techniques are

- widely used for
  - audio/video data
  - simulation on elementary geometry (or images thereof)
- rarely used for
  - geometric modeling
  - simulation on general geometry

Reason (amongst others): approximation is

- well understood on boxes
- partially understood on triangulations
- not well understood on arbitrary domains

Motivation

Problems with TP-splines:

- boundary data
  - weight function

Motivation

Problems with TP-splines:

- boundary data
  - weight function
- stable bases
  - extension
  - normalization

Motivation

Problems with TP-splines:

- boundary data
  - weight function
- stable bases
  - extension
  - normalization
- theoretical results
  - Dahmen, DeVore, Scherer 1980:

$$\min_s \|f - s\|_{\Omega,p} \le C \sum_{i=1}^{d} h_i^{n_i} \|\partial_i^{n_i} f\|_{\Omega,p}$$

Motivation

Problems with TP-splines:

- boundary data
  - weight function
- stable bases
  - extension
  - normalization
- theoretical results
  - Dahmen, DeVore, Scherer 1980:

$$\min_s \|f - s\|_{\Omega,p} \le C \sum_{i=1}^{d} h_i^{n_i} \|\partial_i^{n_i} f\|_{\Omega,p}$$

  - Ω coordinate-wise convex and ...
  - C depends on aspect ratio of grid cells.

Outline

- Interpolation with polynomials on TP grids
- Approximation with polynomials on domains
- Approximation with TP splines on donains

Polynomial interpolation

Theorem

Let $\Gamma = [\gamma_1, \dots, \gamma_n]$ *be a sequence of interpolation nodes in* $\Omega := [0,1]$. *For* $f \in C^n(\Omega)$, *the interpolation problem*

$$p \in \mathbb{P}_n(\mathbb{R}) : p(\Gamma) = f(\Gamma)$$

is uniquely solvable, and the error satisfies

$$f(x) - p(x) = \frac{(x-\gamma_1)\cdots(x-\gamma_n)}{n!} D^n f(\xi), \quad x, \xi \in [0,1].$$

- universal result, covering also Hermite and Taylor case
- elementary proof

Polynomial interpolation

Theorem

Let $\Gamma = [\gamma_1, \ldots, \gamma_n]$ *be a sequence of interpolation nodes in* $\Omega := [0,1]$*.*
For $f \in C^n(\Omega)$*, the interpolation problem*

$$p \in \mathbb{P}_n(\mathbb{R}) : p(\Gamma) = f(\Gamma)$$

is uniquely solvable, and the error satisfies

$$\|f - p\|_{\Omega,\infty} \le C(\Gamma) \left\|D^n f\right\|_{\Omega,\infty}, \quad C(\Gamma) := \frac{\|(\cdot - \gamma_1)\cdots(\cdot - \gamma_n)\|_{\Omega,\infty}}{n!} \le \frac{1}{n!}.$$

- universal result, covering also Hermite and Taylor case
- elementary proof

Polynomial interpolation

Observation

Let $\Gamma = [\gamma_1, \ldots, \gamma_N]$ be a sequence of interpolation nodes in $\Omega := [0, 1]^d$. For $f \in C^n(\Omega)$, the interpolation problem

$$p \in \mathbb{P}_n(\mathbb{R}^d) : p(\Gamma) = f(\Gamma)$$

is not always uniquely solvable, and error representations are complicated.

For polynomials of total order $n \in \mathbb{N}$: Sauer-Xu formula (1995)

Polynomial interpolation

Observation

Let $\Gamma = [\gamma_1, \dots, \gamma_N]$ be a sequence of interpolation nodes in $\Omega := [0,1]^d$. For $f \in C^n(\Omega)$, the interpolation problem

$$p \in \mathbb{P}_n(\mathbb{R}^d) : p(\Gamma) = f(\Gamma)$$

is not always uniquely solvable, and error representations are complicated.

For polynomials of coordinate order $n \in \mathbb{N}^d$ and tensor product grid: de Boor formula (1997)

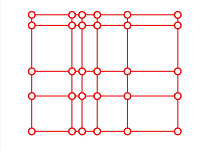
$$f(\mathbf{x}) - p(\mathbf{x}) = \sum_{s=1}^{d} \psi_{s,\Gamma}(\mathbf{x})\big(I_{\Gamma,\backslash s}[((t_{i,s}|_s\cdot) : i = 0, \dots, k_s), \mathbf{x}|\mathbf{i}_s, \dots, \mathbf{i}_s]f\big)(\mathbf{x}_{\backslash s})$$

"... which is an attempt to avoid more cumbersome notation."

Polynomial interpolation on TP grid

Theorem

Let $\Gamma := \Gamma_1 \times \cdots \times \Gamma_d \subset \Omega := [0,1]^d$ be a tensor product grid of dimension $\mathbf{n} = [n_1, \ldots, n_d]$



- *The interpolation problem*

$$p \in \mathbb{P}_{\mathbf{n}} : p(\Gamma) = f(\Gamma)$$

  is uniquely solvable via factorization.

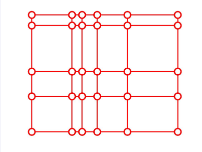
- *If $\partial^\alpha f$ is bounded for all $\alpha \le \mathbf{n}$, then the error satisfies*

$$\|f - p\|_{\infty,\Omega} \le \sum_{\alpha_i \in \{0, n_i\}} C^{\frac{\alpha_1}{n_1}}(\Gamma_1) \cdots C^{\frac{\alpha_d}{n_d}}(\Gamma_d)\, \|\partial^\alpha f\|_{\infty,\Omega}.$$

Polynomial interpolation on TP grid

Theorem

Let $\Gamma := \Gamma_1 \times \cdots \times \Gamma_d \subset \Omega := [0,1]^d$ *be a tensor product grid of dimension* $\mathbf{n} = [n_1, \ldots, n_d]$



- *The interpolation problem*

$$p \in \mathbb{P}_{\mathbf{n}} : p(\Gamma) = f(\Gamma)$$

  is uniquely solvable via factorization.
- *If* $\partial^\alpha f$ *is bounded for all* $\alpha \leq \mathbf{n}$, *then the error satisfies*

$$\|f - p\|_{\infty,\Omega} \leq \sum_{\alpha_i \in \{0, n_i\}} \frac{1}{\alpha!} \|\partial^\alpha f\|_{\infty,\Omega}.$$

Polynomial interpolation on TP grid

Example

For $d = 2$ and $\mathbf{n} = (4, 3)$, we obtain the estimate

$$\|f - p\| \leq \frac{\|\partial_1^4 f\|}{4!} + \frac{\|\partial_2^3 f\|}{3!} + \frac{\|\partial_1^4 \partial_2^3 f\|}{4!\,3!}.$$

Polynomial interpolation on TP grid

Example

For $d = 2$ and $\mathbf{n} = (4,3)$, we obtain the estimate

$$\|f - p\| \le \frac{\|\partial_1^4 f\|}{4!} + \frac{\|\partial_2^3 f\|}{3!} + \frac{\|\partial_1^4 \partial_2^3 f\|}{4!\,3!}.$$

Proof

With $E : f \mapsto f - p$ the error operator, write

$$E = -\sum_{\alpha_i \in \{0,1\}} (-E_d)^{\alpha_d} \cdots (-E_1)^{\alpha_1}$$

and use commutation property

$$\partial_i E_j = E_j \partial_i, \quad i \ne j.$$

Polynomial interpolation on TP grid

Definition

The least distance δ of a sorted sequence $\gamma_1 \leq \cdots \leq \gamma_n$ is defined by

$$\delta := \min_i \gamma_{i+1} - \gamma_i.$$

Theorem (Atkinson 1994, Mößner & R. 2009)

For a TP grid with least distances $\delta_1, \ldots, \delta_d$, the interpolation error is

$$\|f - p\|_{\infty,\Omega} \leq \sum_{j=1}^{d} L_j \, \|\partial_j^{n_j} f\|_{\infty,\Omega}, \qquad L_j := C(\Gamma_j)\, \|I_{j+1}\| \cdots \|I_d\|,$$

where $\|I_j\|$ is the Lebesgue constant of the univariate interpolation operator I_j on Γ_j. Note that

$$C(\Gamma_j) \leq 1/n_j! \quad \textit{and} \quad \|I_j\| \leq n_j \, \delta_j^{1-n_j}.$$

Polynomial interpolation on TP grid

Example

For $d = 2$, $\mathbf{n} = (4,3)$, and least distances δ_1, δ_2, we obtain the estimate

$$\|f - p\| \leq \frac{3}{4!\delta_2^2} \|\partial_1^4 f\| + \frac{1}{3!} \|\partial_2^3 f\|,$$

Polynomial interpolation on TP grid

Example

For $d = 2$, $\mathbf{n} = (4, 3)$, and least distances δ_1, δ_2, we obtain the estimate

$$\|f - p\| \leq \frac{3}{4!\delta_2^2} \|\partial_1^4 f\| + \frac{1}{3!} \|\partial_2^3 f\|,$$

and equally

$$\|f - p\| \leq \frac{4}{3!\delta_1^3} \|\partial_2^3 f\| + \frac{1}{4!} \|\partial_1^4 f\|.$$

Polynomial interpolation on TP grid

Example

For $d = 2$, $\mathbf{n} = (4, 3)$, and least distances δ_1, δ_2, we obtain the estimate

$$\|f - p\| \leq \frac{3}{4!\delta_2^2}\,\|\partial_1^4 f\| + \frac{1}{3!}\,\|\partial_2^3 f\|,$$

and equally

$$\|f - p\| \leq \frac{4}{3!\delta_1^3}\,\|\partial_2^3 f\| + \frac{1}{4!}\,\|\partial_1^4 f\|.$$

Proof

Let $I_* := I_{d-1} \cdots I_1$ and $E_* := \mathrm{Id} - I_*$. Write

$$E = E_d + I_d E_*,$$

and proceed by induction on d.

Polynomial interpolation on TP grid

Chebyshev nodes

For a TP grid with least distances $\delta_1, \ldots, \delta_d$, the interpolation error is

$$\|f - p\|_{\infty,\Omega} \leq \sum_{j=1}^{d} L_j \, \|\partial_j^{n_j} f\|_{\infty,\Omega}, \qquad L_j := C(\Gamma_j)\,\|I_{j+1}\| \cdots \|I_d\|,$$

Chebyshev nodes $\gamma_{r,j} := \cos^2\left(\frac{(2j-1)\pi}{4n_j}\right)$ provide close-to-optimal error bounds,

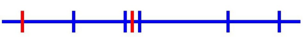
$$\|f - p\| \leq \sum_{j=1}^{d} \frac{2^{2d+1}}{(n_j - 1)!4^{n_j+j}} \, \|\partial_j^{n_j} f\|.$$

Polynomial interpolation on TP grid

Definition

The least distance $\delta[m]$ skipping m nodes is defined by

$$\delta[m] := \min_i \gamma_{i+1+m} - \gamma_i .$$

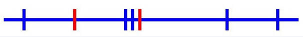


Polynomial interpolation on TP grid

Definition

The least distance $\delta[m]$ skipping m nodes is defined by

$$\delta[m] := \min_i \gamma_{i+1+m} - \gamma_i .$$

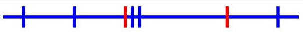


Polynomial interpolation on TP grid

Definition

The least distance $\delta[m]$ skipping m nodes is defined by

$$\delta[m] := \min_i \gamma_{i+1+m} - \gamma_i.$$

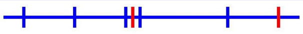


Polynomial interpolation on TP grid

Definition

The least distance $\delta[m]$ skipping m nodes is defined by

$$\delta[m] := \min_i \gamma_{i+1+m} - \gamma_i .$$

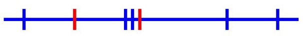


Polynomial interpolation on TP grid

Definition

The least distance $\delta[m]$ skipping m nodes is defined by

$$\delta[m] := \min_i \gamma_{i+1+m} - \gamma_i.$$



Theorem (Mößner & R. 2009)

Let $\mathbf{m} \in \mathbb{N}_0^d$ *be chosen such that*

$$\langle \mathbf{m}, \mathbf{n}^{-1} \rangle := \frac{m_1}{n_1} + \cdots + \frac{m_d}{n_d} < 1.$$

For a TP grid with least distances $\delta_1[m_1], \ldots, \delta_d[m_d]$ *bounded by* $\delta_r[m_r] \geq \bar{\delta} > 0$, *the interpolation error is bounded by*

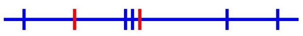
$$\|f - p\|_{\infty,\Omega} \leq c(\bar{\delta}) \sum_{j=1}^{d} \|\partial_j^{n_j} f\|_{\infty,\Omega}.$$

Polynomial interpolation on TP grid

Definition

The least distance $\delta[m]$ skipping m nodes is defined by

$$\delta[m] := \min_i \gamma_{i+1+m} - \gamma_i .$$



Theorem (Mößner & R. 2009)

Let $\mathbf{m} \in \mathbb{N}_0^d$ *be chosen such that*

$$\langle \mathbf{m}, \mathbf{n}^{-1} \rangle := \frac{m_1}{n_1} + \cdots + \frac{m_d}{n_d} < 1.$$

For a TP grid with least distances $\delta_1[m_1], \ldots, \delta_d[m_d]$ *bounded by* $\delta_r[m_r] \geq \bar{\delta} > 0$, *the interpolation error is bounded by*

$$\|\partial^\alpha (f - p)\|_{\infty,\Omega} \leq c(\bar{\delta}) \sum_{j=1}^{d} \|\partial_j^{n_j} f\|_{\infty,\Omega}, \quad \langle \alpha, \mathbf{n}^{-1} \rangle < 1.$$

Polynomial interpolation on TP grid

Proof

Use embedding theorem in anisotropic Sobolev spaces,

$$\|\partial^\alpha f\|_\infty \le c_1\Big(\|f\|_\infty + \sum_{j=1}^d \|\partial_j^{n_j} f\|_\infty\Big), \quad \frac{\alpha_1}{n_1} + \cdots + \frac{\alpha_d}{n_d} < 1,$$

and the mean value theorem for divided differences to show that

$$\|I_r f\|_\infty \le c_2(\delta_r)\Big(\|f\|_\infty + \|\partial_r^{n_r} f\|_\infty\Big).$$

Example

Consider $f(x, y) := (x+y)^{11/2}$, $(x, y) \in [0,1]^2$ and order $\mathbf{n} = (5,5)$:

The estimate

$$\|f - p\| \leq \frac{\|\partial_1^5 f\|}{5!} + \frac{\|\partial_2^5 f\|}{5!} + \frac{\|\partial_1^5 \partial_2^5 f\|}{5! \cdot 5!}$$

does not apply since $\partial_1^5 \partial_2^5 f$ is unbounded.

Example

Consider $f(x,y) := (x+y)^{11/2}$, $(x,y) \in [0,1]^2$ and order $\mathbf{n} = (5,5)$:

For $\Gamma_1 = \Gamma_2 = [0, \varepsilon, 2\varepsilon, 3\varepsilon, 1]$, the interpolation error is unbounded,

$$\|f - p\| \geq \frac{1}{50\sqrt{\varepsilon}}.$$

- For $\mathbf{m} = (2,2)$, we have

$$\langle \mathbf{m}, \mathbf{n}^{-1} \rangle = 4/5 < 1 \quad \text{but} \quad \delta_1[2] = \delta_2[2] = 1/(3\varepsilon).$$

- For $\mathbf{m} = (3,3)$, we have

$$\delta_1[3] = \delta_2[3] = 1 \quad \text{but} \quad \langle \mathbf{m}, \mathbf{n}^{-1} \rangle = 6/5 > 1.$$

Example

Consider $f(x,y) := (x+y)^{11/2}$, $(x,y) \in [0,1]^2$ and order $\mathbf{n} = (5,5)$:

For $\Gamma_1 = [0,0,0,0,1]$ and $\Gamma_2 = [0,0,1/2,1/2,1]$, the interpolation error is bounded,

$$\|f - p\| \approx 0.32.$$

Indeed, for $\mathbf{m} = (3,1)$, we have

$$\langle \mathbf{m}, \mathbf{n}^{-1} \rangle = 4/5 < 1 \quad \text{and} \quad \delta_1[3] = 1,\ \delta_2[1] = 1/2.$$

Approximation power of polynomials

Fundamental question

Given a function f defined on some domain $\Omega \subset \mathbb{R}^d$, and a subspace $\mathbb{P}_*$ of polynomials, how well can f be approximated by a polynomial $p \in \mathbb{P}_*$?

Approximation power of polynomials

Fundamental question

Given a function f defined on some domain $\Omega \subset \mathbb{R}^d$, and a subspace $\mathbb{P}_*$ of polynomials, how well can f be approximated by a polynomial $p \in \mathbb{P}_*$?

The answer will depend on

- the subspace of polynomials
- the regularity of f
- the geometry of Ω

Bramble-Hilbert Lemma

Theorem (Bramble and Hilbert 1971)

Let $\Omega \subset [0,1]^d$ be a connected domain with Lipschitz boundary.

For any function $f \in W_p^n(\Omega)$ there exists a polynomial $\pi \in \mathbb{P}_n$ of total order n such that

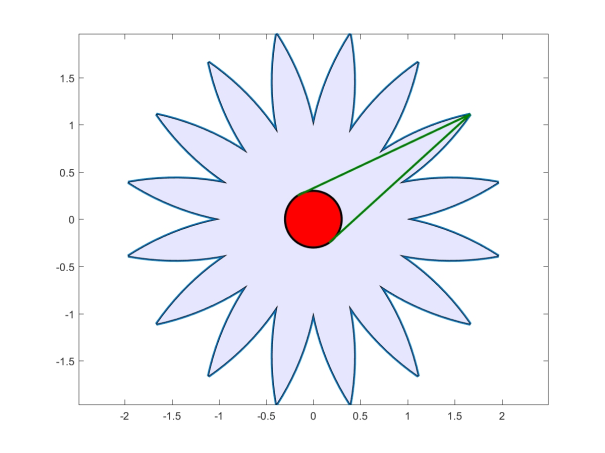
$$\|f - \pi\| \leq c(\Omega) \sum_{|\alpha|=n} \|\partial^\alpha f\|_{L^p(\Omega)}.$$

- Main work done by Morrey (1966).
- Proof non-constructive.
- Dependence of constant on the shape of Ω not specified .

Bramble-Hilbert Lemma

Theorem (Dupont and Scott 1980)

Let $\Omega \subset [0,1]^d$ be star-shaped wrt. some ball.



Bramble-Hilbert Lemma

Theorem (Dupont and Scott 1980)

Let $\Omega \subset [0,1]^d$ be star-shaped wrt. some ball.

For any function $f \in W_p^n(\Omega)$ there exists a polynomial $\pi \in \mathbb{P}_n$ of total order n such that

$$\|f - \pi\| \le c(\Omega) \sum_{|\alpha|=n} \|\partial^\alpha f\|_{L^p(\Omega)}.$$

- π defined as averaged Taylor polynomial.
- Similar proof technique used for Sobolev embedding theorems.
- Dependence of constant on the shape of Ω not specified explicitly.

Bramble-Hilbert Lemma

Theorem (Durán 1983, Dechevski and Wendland 2006)

Let $\Omega \subset [0,1]^d$ be star-shaped wrt. some ball.

For any function $f \in W_p^n(\Omega)$ there exists a polynomial $\pi \in \mathbb{P}_n$ of total order n such that

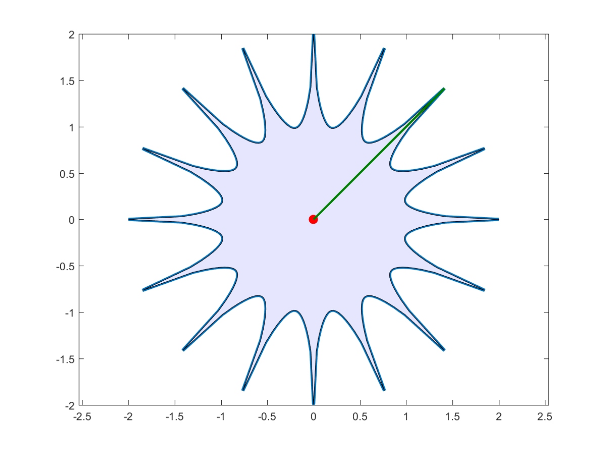
$$\|f - \pi\| \leq c(\Omega) \sum_{|\alpha|=n} \|\partial^\alpha f\|_{L^p(\Omega)}.$$

- π defined as averaged Taylor polynomial.
- Dependence of constant on the shape of Ω specified explicitly.

Bramble-Hilbert Lemma

Theorem (Dechevski and Quak 1990)

Let $\Omega \subset [0,1]^d$ be star-shaped wrt. a single point.



Bramble-Hilbert Lemma

Theorem (Dechevski and Quak 1990)

Let $\Omega \subset [0,1]^d$ be star-shaped wrt. a single point.

For any function $f \in W_p^n(\Omega)$ there exists a polynomial $\pi \in \mathbb{P}_n$ of total order n such that

$$\|f - \pi\| \leq c(\Omega) \sum_{|\alpha|=n} \|\partial^\alpha f\|_{L^p(\Omega)}, \quad p > d.$$

- π defined as averaged Taylor polynomial.
- Dependence of constant on the shape of Ω specified explicitly.

Bramble-Hilbert Lemma

Theorem (Verfürth 1999)

Let $\Omega \subset [0,1]^d$ be star-shaped wrt. a single point.

For any function $f \in W_p^n(\Omega)$ there exists a polynomial $\pi \in \mathbb{P}_n$ of total order n such that

$$\|f - \pi\| \leq c(\Omega) \sum_{|\alpha|=n} \|\partial^\alpha f\|_{L^p(\Omega)}, \quad p \geq 2.$$

- Based on Poincaré inequality.
- π defined by mean value interpolation.
- Dependence of constant on the shape of Ω specified explicitly.

Bramble-Hilbert Lemma

Theorem (Thrun 2003)

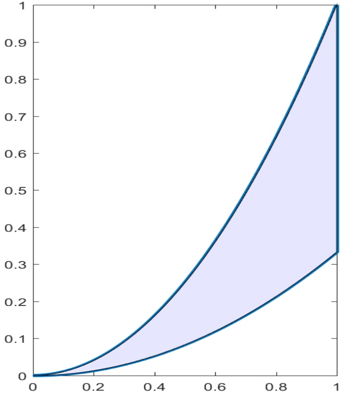
Let $\Omega \subset [0,1]^d$ be star-shaped wrt. a single point.

For any function $f \in W_p^n(\Omega)$ there exists a polynomial $\pi \in \mathbb{P}_n$ of total order n such that

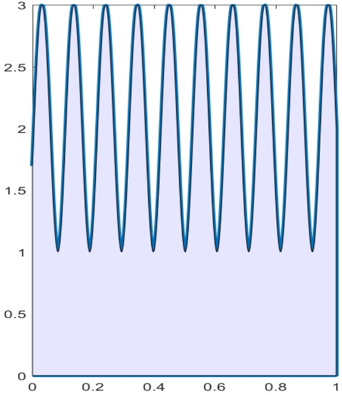
$$\|f - \pi\| \le c(\Omega) \sum_{|\alpha|=n} \|\partial^\alpha f\|_{L^p(\Omega)}, \quad p \ge 1.$$

- Based on Poincaré inequality.
- π defined by mean value interpolation.
- Dependence of constant on the shape of Ω specified explicitly.

Bramble-Hilbert Lemma



not covered



diverging constant

Bramble-Hilbert Lemma

Theorem (R. 2009)

Let $\Omega \subset [0,1]^d$

- *be bounded by a finite number of axis-aligned graphs,*
- *contain some box* B.

Given $f \in W_p^{\mathbf{n}}(\Omega)$, *define the polynomial* $\pi \in \mathbb{P}_{\mathbf{n}}$ *of coordinate order* $\mathbf{n} \in \mathbb{N}^d$ *by*

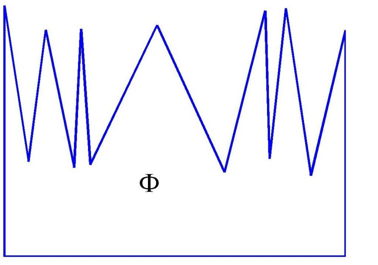
$$\pi := \sum_{\alpha < \mathbf{n}} \langle f, q_\alpha \rangle_B q_\alpha,$$

where the q_α *are tensor product Legendre polynomials on* B. *Then*

$$\|f - \pi\|_{L^p(\Omega)} \leq \sum_{j=1}^{d} c_j(\Omega, \mathbf{n}) \left\| \partial_j^{n_j} f \right\|_{L^p(\Omega)}, \quad p \geq 1.$$

Bramble-Hilbert Lemma

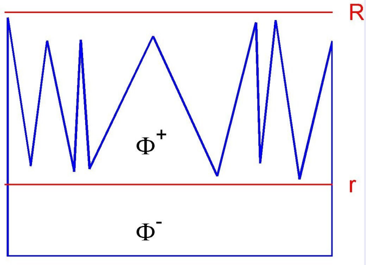
Lemma



Let $Y \subset [0,1]^{d-1}$ be measurable, and $\varphi : Y \to [r, R]$ be continuous.

Bramble-Hilbert Lemma

Lemma



Let $Y \subset [0,1]^{d-1}$ be measurable, and $\varphi : Y \to [r, R]$ be continuous. Then

$$\|\Delta\|_{L^p(\Phi^+)} \leq \gamma_1 \|\Delta\|_{L^p(\Phi^-)} + \gamma_2 \|\partial_d^{n_d} \Delta\|_{L^p(\Phi)},$$

where the constants γ_1, γ_2 depend only on R/r, and $\mathbf{n}, p$.

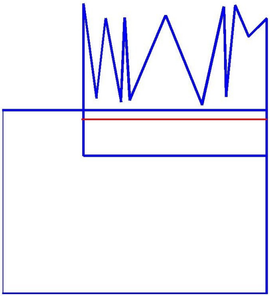
Bramble-Hilbert Lemma

Recursive construction of domain Ω:

- Start with some box $\Omega_0 := B \subset \mathbb{R}^d$,

Bramble-Hilbert Lemma

Recursive construction of domain Ω:

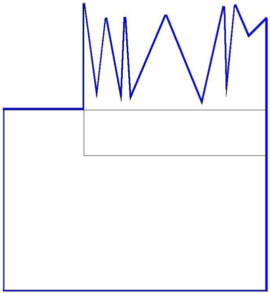


- Start with some box $\Omega_0 := B \subset \mathbb{R}^d$,
- For $j = 1, \ldots, J$ do: Choose some graph domain Φ_j and an axis-aligned isometry I_j such that

$$I_j(\Phi_j^-) \subset \Omega_{j-1} \quad \text{and set} \quad \Omega_j := \Omega_{j-1} \cup I_j(\Phi_j).$$

Bramble-Hilbert Lemma

Recursive construction of domain Ω:

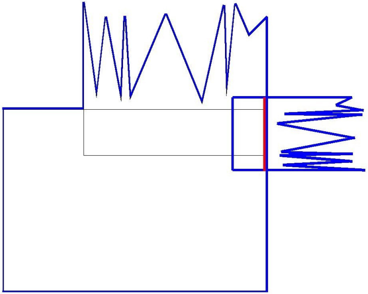


- Start with some box $\Omega_0 := B \subset \mathbb{R}^d$,
- For $j = 1, \ldots, J$ do: Choose some graph domain Φ_j and an axis-aligned isometry I_j such that

$$I_j(\Phi_j^-) \subset \Omega_{j-1} \quad \text{and set} \quad \Omega_j := \Omega_{j-1} \cup I_j(\Phi_j).$$

Bramble-Hilbert Lemma

Recursive construction of domain Ω:



- Start with some box $\Omega_0 := B \subset \mathbb{R}^d$,
- For $j = 1, \ldots, J$ do: Choose some graph domain Φ_j and an axis-aligned isometry I_j such that

$$I_j(\Phi_j^-) \subset \Omega_{j-1} \quad \text{and set} \quad \Omega_j := \Omega_{j-1} \cup I_j(\Phi_j).$$

- Set $\Omega := \Omega_J$

Bramble-Hilbert Lemma

Recursive construction of domain Ω:

- Start with some box $\Omega_0 := B \subset \mathbb{R}^d$,
- For $j = 1, \ldots, J$ do: Choose some graph domain Φ_j and an axis-aligned isometry I_j such that

$$I_j(\Phi_j^-) \subset \Omega_{j-1} \quad \text{and set} \quad \Omega_j := \Omega_{j-1} \cup I_j(\Phi_j).$$

- Set $\Omega := \Omega_J$

Proof of Theorem:

- Error $\Delta := f - \pi$ is small on box Ω_0 by construction.
- Control error propagation $\|\Delta\|_{L^p(\Omega_{j-1})} \to \|\Delta\|_{L^p(\Omega_j)}$ by means of the lemma.

Poincaré inequality

Theorem (R. 2009)

Let $\Omega \subset [0,1]^d$

- *be bounded by a finite number of axis-aligned graphs,*
- *contain some box* B.

Given $f \in W_p^1(\Omega)$, *define the constant* $\pi \in \mathbb{P}_1$ *by*

$$\pi := \frac{1}{\operatorname{vol} B} \int_B f,$$

Then

$$\|f - \pi\|_{L^p(\Omega)} \le c(\Omega)\, \|\nabla f\|_{L^p(\Omega)}, \quad p \ge 1.$$

Poincaré inequality

Theorem (R. 2009)

Let $\Omega \subset [0,1]^d$

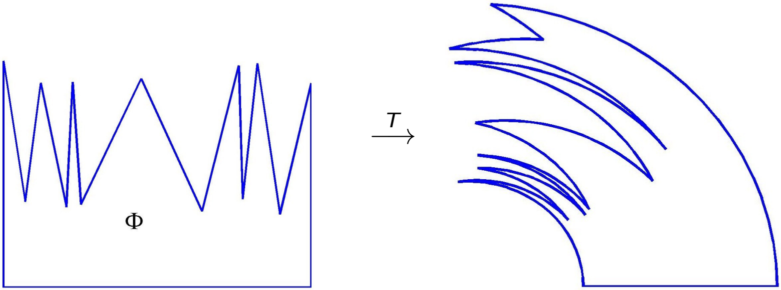
- *be bounded by a finite number of diffeomorphic images of graphs,*
- *contain some box* B.

Given $f \in W_p^1(\Omega)$, *define the constant* $\pi \in \mathbb{P}_1$ *by*

$$\pi := \frac{1}{\mathrm{vol}\, B} \int_B f,$$

Then

$$\|f - \pi\|_{L^p(\Omega)} \leq c(\Omega)\, \|\nabla f\|_{L^p(\Omega)}, \quad p \geq 1.$$


Φ
T

Lemma

Let

- Φ, Φ^-, Φ^+ be defined as before,
- $T : \Phi \to \mathbb{R}^d$ be a diffeomorphism.

Then

$$\|\Delta\|_{L^p(T(\Phi^+))} \leq \gamma_1 \|\Delta\|_{(L^p(T(\Phi^-))} + \gamma_2 \|\nabla\Delta\|_{L^p(T(\Phi))},$$

where the constants γ_1, γ_2

- depend only on r, R and $\operatorname{cond}(T) := \|DT\|_\infty \cdot \|DT^{-1}\|_\infty$,
- can be computed explicitly.

Poincaré inequality

Recursive construction of Ω:

- Start with some box $\Omega_0 := B \subset \mathbb{R}^d$,
- For $j = 1, \ldots, J$ do: Choose some φ_j-domain Φ_j and a diffeomorphism T_j such that

$$T_j(\Phi_j^-) \subset \Omega_{j-1} \quad \text{and set} \quad \Omega_j := \Omega_{j-1} \cup I_j(\Phi_j).$$

- Set $\Omega := \Omega_J$

Proof of Theorem:

- Error $\Delta := f - \pi$ is small on box Ω_0 by construction.
- Control error propagation $\|\Delta\|_{L^p(\Omega_{j-1})} \to \|\Delta\|_{L^p(\Omega_j)}$ by means of the lemma.

Bramble-Hilbert Lemma

Theorem (R. 2009)

Let $\Omega \subset [0,1]^d$

- *be bounded by a finite number of diffeomorphic images of graphs,*
- *contain some box* B.

Given $f \in W_p^n(\Omega)$, *define the polynomial* $\pi \in \mathbb{P}_n$ *of total order* n *by*

$$\int_B \partial^\alpha f = \int_B \partial^\alpha \pi, \quad |\alpha| < n.$$

Then

$$\|f - \pi\|_{L^p(\Omega)} \leq c(\Omega, n)\, |f|_{W_p^n(\Omega)}, \quad p \geq 1,\ |\alpha| \leq n.$$

Bramble-Hilbert Lemma

Theorem (R. 2009)

Let $\Omega \subset [0,1]^d$

- *be bounded by a finite number of diffeomorphic images of graphs,*
- *contain some box* B.

Given $f \in W_p^n(\Omega)$, *define the polynomial* $\pi \in \mathbb{P}_n$ *of total order* n *by*

$$\int_B \partial^\alpha f = \int_B \partial^\alpha \pi, \quad |\alpha| < n.$$

Then

$$\|\partial^\alpha (f - \pi)\|_{L^p(\Omega)} \leq c(\Omega, n)\, |f|_{W_p^n(\Omega)}, \quad p \geq 1,\ |\alpha| \leq n.$$

Bramble-Hilbert Lemma

Generalizations:

- Let $P : W_p^n(\Omega) \to \mathbb{P}_n$ be a Hölder-continuous projector,

$$\|P(f) - P(g)\|_{L^p(\Omega)} \le L\,\|f - g\|_{L^p(\Omega)}^s, \quad 0 < s \le 1.$$

Then

$$\|\partial^\alpha (f - P(f))\|_{L^p(\Omega)} \le c(\Omega, n)\,|f|_{W_p^n(\Omega)}^s, \quad p \ge 1,\ |\alpha| \le n.$$

Bramble-Hilbert Lemma

Generalizations:

- Let $P : W_p^n(\Omega) \to \mathbb{P}_n$ be a Hölder-continuous projector,

$$\|P(f) - P(g)\|_{L^p(\Omega)} \le L\, \|f - g\|_{L^p(\Omega)}^s, \quad 0 < s \le 1.$$

  Then

$$\|\partial^\alpha (f - P(f))\|_{L^p(\Omega)} \le c(\Omega, n)\, |f|_{W_p^n(\Omega)}^s, \quad p \ge 1,\ |\alpha| \le n.$$

- Replace the total order spaces $\mathbb{P}_n$ by any D-invariant subspace $\mathbb{P}_*$ of polynomials.

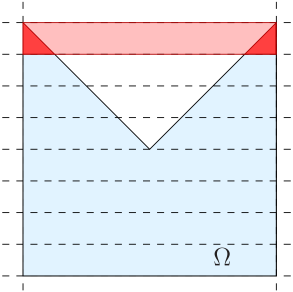
Spline approximation

The Dahmen-DeVore-Scherer error estimate

$$\min_{s} \|f - s\|_{\Omega,p} \leq C \sum_{i=1}^{d} h_i^{n_i} \|\partial_i^{n_i} f\|_{\Omega,p}$$

leaves many questions open.

2d example



- Observation: constant C in error estimate depends on h_1/h_2.
- Reason: B-splines with disconnected support.
- Idea: The space of diversified B-splines contains a seperate copy of B_i for each connected component of supp $B_i \cap \Omega$.

Diversified B-Splines

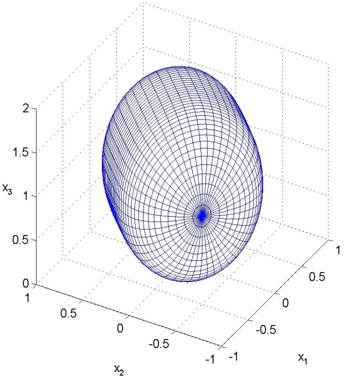
Theorem (R., Sissouno '14, '15)

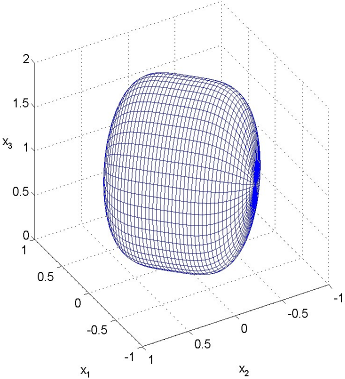
Let $\Omega \subset \mathbb{R}^2$ be a domain bounded by a finite number of axis-aligned Lipschitz graphs. There exists a constant C depending only on the order $\mathbf{n}$ and Ω with

$$\inf_{s \in \mathcal{S}^*} \|f - s\|_p \leq C \sum_{i=1}^{2} h_i^{n_i} \|\partial_i^{n_i} f\|_p$$

for any $f \in W_p^{\mathbf{n}}(\Omega)$, where $\mathcal{S}^*$ is a space of diversifed B-splines of order $\mathbf{n}$ and knots with maximal spacing h_1, h_2.

A counterexample in 3d





- Domain $\Omega := \{\mathbf{x} : (x_1 - x_2)^2 + (x_1 + x_2)^4 + (1 - x_3)^2 < 1\}$.
- Order $\mathbf{n} = (n, n, n)$, where $n \geq 2$. .
- Knots $T_1 := T_2 := h\mathbb{Z}$, $T_3 := h^5\mathbb{Z}$.
- Function $f_h(x_1, x_2, x_3) := x_2^{2n-2}((2n-1)x_1 - (n-1)x_2)\exp(-x_3/h^4)$.
- Error $\|f_h - s\|_\infty \geq \frac{C}{h+h^4} \sum_{i=1}^{3} h_i^{n_i} \|\partial_i^{n_i} f_h\|_\infty$.

Conclusion

A better understanding of approximation on domains

- will narrow the gap between multivariate approximation and applications,
- requires more basic research.